

Nesterenko's linear independence criterion for vectors

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Abstract

In this paper we deduce a lower bound for the rank of a family of p vectors in \mathbb{R}^k (considered as a vector space over the rationals) from the existence of a sequence of linear forms on \mathbb{R}^p , with integer coefficients, which are small at k points. This is a generalization to vectors of Nesterenko's linear independence criterion (which corresponds to $k = 1$). It enables one to make use of some known constructions of linear forms small at several points, related to Padé approximation. As an application, we prove that at least $\frac{2\log a}{1+\log 2}(1+o(1))$ odd integers $i \in \{3, 5, \dots, a\}$ are such that either $\zeta(i)$ or $\zeta(i+2)$ is irrational, where a is an odd integer, $a \rightarrow \infty$. This refines upon Ball-Rivoal's theorem, namely $\zeta(i) \notin \mathbb{Q}$ for at least $\frac{\log a}{1+\log 2}(1+o(1))$ such i .

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1 Introduction

The motivation for this paper comes from irrationality results on values of Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ at odd integers $s \geq 3$. The first result is due to Apéry [1]: $\zeta(3) \notin \mathbb{Q}$. The next breakthrough in this topic is due to Rivoal [23] and Ball-Rivoal [2]:

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \zeta(7), \dots, \zeta(a)) \geq \frac{\log a}{1 + \log 2}(1 + o(1)) \quad (1.1)$$

as $a \rightarrow \infty$, where a is an odd integer; notice this is a lower bound on the rank of this family of real numbers, in \mathbb{R} considered as a vector space over the rationals. Conjecturally the left handside is equal to $\frac{a+1}{2}$, but even the constant $\frac{1}{1+\log 2}$ in Eq. (1.1) has never been improved. Actually, known refinements of Ball-Rivoal's proof provide new results only for fixed values of a : the improvement always lies inside the error term $o(1)$ as $a \rightarrow \infty$.

In this text we prove the following result, which is a strict improvement upon Eq. (1.1).

Theorem 1. *In \mathbb{R}^2 seen as a \mathbb{Q} -vector space, the family of vectors*

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \zeta(3) \\ 6\zeta(5) \end{pmatrix}, \begin{pmatrix} \zeta(5) \\ 15\zeta(7) \end{pmatrix}, \dots, \begin{pmatrix} \zeta(a) \\ \frac{a(a+1)}{2}\zeta(a+2) \end{pmatrix}$$

has rank greater than or equal to $\frac{2\log a}{1+\log 2}(1+o(1))$ as $a \rightarrow \infty$, with a odd.

It is clear that Theorem 1 implies Eq. (1.1): if E is a \mathbb{Q} -subspace of \mathbb{R} which contains 1, $\zeta(3), \zeta(5), \dots, \zeta(a)$ then $E \times E$ contains all vectors of Theorem 1. Using linear algebra it is not difficult (see §2.3) to prove that Theorem 1 is a strict improvement upon Ball-Rivoal's result (1.1), namely it provides more information of $\zeta(3), \dots, \zeta(a)$.

Theorem 1 can also be stated as follows (see §2.3): if F is a subspace of $\mathbb{R}^{(a+3)/2}$, defined over \mathbb{Q} , which contains both points $e_1 = (1, 0, \zeta(3), \zeta(5), \dots, \zeta(a))$ and $e_2 = (0, 1, \binom{4}{2}\zeta(5), \binom{6}{2}\zeta(7), \dots, \binom{a+1}{2}\zeta(a+2))$, then $\dim_{\mathbb{R}} F \geq \frac{2\log a}{1+\log 2}(1+o(1))$.

Among the vectors $\begin{pmatrix} \zeta(i) \\ \binom{i+1}{2}\zeta(i+2) \end{pmatrix}$ of Theorem 1, we may discard those such that both $\zeta(i)$ and $\zeta(i+2)$ are rational numbers. Accordingly there are at least $\frac{2\log a}{1+\log 2}(1+o(1))$ odd integers $i \in \{3, 5, \dots, a\}$ such that either $\zeta(i)$ or $\zeta(i+2)$ is irrational. This consequence of Theorem 1 can be stated as follows (see §4.2).

Corollary 1. *Let $a \geq 3$ be an odd integer. Denote by $n_1(a)$ the number of odd integers $i \in \{3, 5, \dots, a\}$ such that $\zeta(i) \notin \mathbb{Q}$, and by $n_2(a)$ the number of odd integers $j \in \{5, 7, \dots, a\}$ such that $\zeta(j) \notin \mathbb{Q}$ and $\zeta(j-2) \in \mathbb{Q}$. Then we have*

$$n_1(a) + n_2(a) \geq \frac{2\log a}{1+\log 2}(1+o(1))$$

as $a \rightarrow \infty$.

Since $n_2(a) \leq n_1(a)$ by definition, this corollary immediately implies the lower bound $n_1(a) \geq \frac{\log a}{1+\log 2}(1+o(1))$ which follows from Ball-Rivoal's result (1.1). In the case where $n_1(a) < \frac{2\log a}{1+\log 2}(1+o(1))$, it asserts also that the set of odd $i \leq a$ such that $\zeta(i) \notin \mathbb{Q}$ does not consist in few large blocks of consecutive odd integers (because $n_2(a)$ is essentially the number of such blocks). For instance, if $n_1(a) = \frac{\log a}{1+\log 2}(1+o(1))$ then only $o(\log a)$ odd integers $j \leq a$ satisfy both $\zeta(j) \notin \mathbb{Q}$ and $\zeta(j-2) \notin \mathbb{Q}$: if there are very few irrational values they have to be mostly non-consecutive. Under the same assumption, there are $\frac{\log a}{1+\log 2}(1+o(1))$ odd integers $j \leq a$ such that $\zeta(j) \notin \mathbb{Q}$ but $\zeta(j-2), \zeta(j+2) \in \mathbb{Q}$: “almost all” blocks consist in just one odd integer.

The linear forms we use to prove Theorem 1 provide also the following result:

Theorem 2. *Let $\lambda \in \mathbb{R}$. When a is odd and $a \rightarrow \infty$, the dimension of the \mathbb{Q} -vector space spanned by the numbers*

$$1, \lambda, \zeta(3) + 6\lambda\zeta(5), \zeta(5) + 15\lambda\zeta(7), \dots, \zeta(a) + \binom{a+1}{2}\lambda\zeta(a+2)$$

is greater than or equal to $\frac{\log a}{1+\log 2}(1+o(1))$.

The proof of Theorems 1 and 2 is based on the following variant of Ball-Rivoal's construction, to which a linear independence criterion is applied: Nesterenko's [21] for Theorem 2, and a generalization to vectors (namely Theorem 3 below) for Theorem 1.

Let a, r, n be positive integers such that a is odd and $6r \leq a$. Consider

$$S_n = (2n)!^{a-6r} \sum_{t=n+1}^{\infty} \frac{(t - (2r+1)n)_{2rn}^3 (t+n+1)_{2rn}^3}{(t-n)_{2n+1}^a}$$

and

$$S_n'' = \frac{1}{2}(2n)!^{a-6r} \sum_{t=n+1}^{\infty} \frac{d^2}{dt^2} \left(\frac{(t - (2r+1)n)_{2rn}^3 (t+n+1)_{2rn}^3}{(t-n)_{2n+1}^a} \right)$$

where Pochhammer's symbol is defined by $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$. The same symmetry argument as in Ball-Rivoal's construction yields

$$S_n = \ell_{0,n} + \ell_{3,n}\zeta(3) + \ell_{5,n}\zeta(5) + \dots + \ell_{a,n}\zeta(a)$$

with $\ell_{0,n}, \ell_{3,n}, \ell_{5,n}, \dots, \ell_{a,n} \in \mathbb{Q}$; the definition of S_n'' (namely like S_n , but with a double derivation) implies

$$S_n'' = \ell_{0,n}'' + \ell_{3,n} \binom{4}{2} \zeta(5) + \ell_{5,n} \binom{6}{2} \zeta(7) + \dots + \ell_{a,n} \binom{a+1}{2} \zeta(a+2)$$

with the same coefficients $\ell_{3,n}, \ell_{5,n}, \dots, \ell_{a,n}$, except for $\ell_{0,n}'' \in \mathbb{Q}$. Therefore the linear form $\ell_{0,n}X_0 + \ell_{0,n}''X_0'' + \ell_{3,n}X_3 + \ell_{5,n}X_5 + \dots + \ell_{a,n}X_a$ is very small at both points $e_1 = (1, 0, \zeta(3), \zeta(5), \dots, \zeta(a))$ and $e_2 = (0, 1, \binom{4}{2}\zeta(5), \binom{6}{2}\zeta(7), \dots, \binom{a+1}{2}\zeta(a+2))$; applying Theorem 3 below yields Theorem 1. The same construction yields Theorem 2 by applying Nesterenko's original linear independence criterion to $S_n + \lambda S_n''$; see §4 for details.

The construction of S_n and S_n'' is analogous to that of [17], where the analogue of Theorem 2 is proved for the numbers $\zeta(j) + j\lambda\zeta(j+1)$ with j odd, $3 \leq j \leq a$. Theorem 3 applies to the construction of [17], but the (qualitative) result it provides is a triviality: since $\zeta_{j+1} \in \mathbb{Q}^* \pi^{j+1}$ because $j+1$ is even, the vectors $\left(\zeta_{j\zeta(j+1)}^{(j)} \right)$ are obviously \mathbb{Q} -linearly independent in \mathbb{R}^2 because their second coordinates are. The situation is exactly the same for Théorème 2 of [10], with the numbers $\text{Li}_j(z) + \lambda \frac{\log^j(z)}{(j-1)!}$, where $z \in \mathbb{Q} \cap (0, 1)$, because $\log z$ is a transcendental number. However it should be emphasized that the quantitative form (iii) of Theorem 3 still provides new information in these settings: it enables one to take advantage really of the properties of the linear forms (see the remarks at the end of this introduction).

More generally, the kind of construction used in [17], [10] and here, namely using a derivation as in the definition of S_n'' to obtain linear forms small at several points, is very classical (especially in connection to simultaneous Padé approximation): see for instance [3] or [13]. This idea is also involved in results of Gutnik and Hessami-Pilehrood stated in §2.4 below. Our main result, namely the following generalization of Nesterenko's linear independence criterion [21] to vectors, applies in these situations.

We let \mathbb{R}^p be endowed with its canonical scalar product and the corresponding norm.

Theorem 3. *Let $1 \leq k \leq p-1$, and $e_1, \dots, e_k \in \mathbb{R}^p$.*

Let $\tau_1, \dots, \tau_k > 0$ be pairwise distinct real numbers.

Let $\omega_1, \dots, \omega_k, \varphi_1, \dots, \varphi_k$ be real numbers such that there exist infinitely many integers n with the following property: for any $j \in \{1, \dots, k\}$, $n\omega_j + \varphi_j \not\equiv \frac{\pi}{2} \pmod{\pi}$.

Let $(Q_n)_{n \geq 1}$ be an increasing sequence of positive integers, such that $Q_{n+1} = Q_n^{1+O(1/n)}$; if $\omega_1 = \dots = \omega_k = 0$, this assumption can be weakened to $Q_{n+1} = Q_n^{1+o(1)}$.

For any $n \geq 1$, let $L_n = \ell_{1,n}X_1 + \dots + \ell_{p,n}X_p$ be a linear form on \mathbb{R}^p , with integer coefficients $\ell_{i,n}$ such that, as $n \rightarrow \infty$:

$$|L_n(e_j)| = Q_n^{-\tau_j+o(1)} |\cos(n\omega_j + \varphi_j) + o(1)| \text{ for any } j \in \{1, \dots, k\}, \quad (1.2)$$

and

$$\max_{1 \leq i \leq p} |\ell_{i,n}| \leq Q_n^{1+o(1)}.$$

Then:

(i) *If F is a subspace of \mathbb{R}^p defined over \mathbb{Q} which contains e_1, \dots, e_k then*

$$\dim F \geq k + \tau_1 + \dots + \tau_k.$$

In other words, letting $C_1, \dots, C_p \in \mathbb{R}^k$ denote the columns of the matrix whose rows are $e_1, \dots, e_k \in \mathbb{R}^p$, we have

$$\text{rk}_{\mathbb{Q}}(C_1, \dots, C_p) \geq k + \tau_1 + \dots + \tau_k$$

in \mathbb{R}^k seen as a \mathbb{Q} -vector space.

(ii) *The vectors e_1, \dots, e_k are \mathbb{R} -linearly independent in \mathbb{R}^p , and the \mathbb{R} -subspace they span does not intersect $\mathbb{Q}^p \setminus \{(0, \dots, 0)\}$.*

(iii) *Let $\varepsilon > 0$, and Q be sufficiently large (in terms of ε). Let \mathcal{C} denote the set of all vectors that can be written as $\lambda_1 e_1 + \dots + \lambda_k e_k + u$ with:*

$$\begin{cases} \lambda_1, \dots, \lambda_k \in \mathbb{R} \text{ such that } |\lambda_j| \leq Q^{\tau_j} \text{ for any } j \in \{1, \dots, k\} \\ u \in (\text{Span}_{\mathbb{R}}(e_1, \dots, e_k))^{\perp} \text{ such that } \|u\| \leq Q^{-1-\varepsilon} \end{cases}$$

Then $\mathcal{C} \cap \mathbb{Z}^p = \{(0, \dots, 0)\}$.

If $k = 1$ and $\omega_1 = \varphi_1 = 0$, this is exactly Nesterenko's linear independence criterion [21]. With $k = 1$, the generalization to arbitrary values of ω_1 and φ_1 has been proved in [8] when $Q_n = \beta^n$ for some $\beta > 1$; this is the most interesting case since it includes oscillating behaviours of the linear forms provided by the saddle point method (and S_n'' defined above has this kind of asymptotics). Note that when $\omega_j = 0$, we assume $\varphi_j \not\equiv \frac{\pi}{2} \pmod{\pi}$ so that Eq. (1.2) reads $|L_n(e_j)| = Q_n^{-\tau_j+o(1)}$. At last, in general Nesterenko's criterion is stated under the assumption $Q_{n+1} = Q_n^{1+o(1)}$; however assuming $Q_{n+1} = Q_n^{1+O(1/n)}$ if some ω_j is

non-zero is not a serious drawback since any sequence $Q_n = \beta^{n^d}$ with $\beta > 1$ and $d > 0$ satisfies this property.

In the conclusions, (ii) is an easy result, and (iii) is the main part (it is a quantitative version of (ii)). We deduce (i) from (iii) using Minkowski's convex body theorem, thereby generalizing the proof given in [12] and [11] of Nesterenko's linear independence criterion. The equivalence between both statements of (i) comes from linear algebra; it is proved in §2.3.

A result analogous to Theorem 3, but in which p linearly independent linear forms like L_n appear in the assumption, is proved in §2.4. This linear independence criterion (in the style of Siegel's) is much easier to prove than Theorem 3. Both results can be thought of as transference principles. In this respect it is worth pointing out that in Theorem 3 we assume essentially that for any positive integer Q there is a linear form : indeed this is L_n , where n is such that $Q_n \leq Q < Q_{n+1}$ so that $Q = Q_n^{1+o(1)}$ because $Q_{n+1} = Q_n^{1+o(1)}$.

The assumption that τ_1, \dots, τ_k are pairwise distinct is very important in Theorem 3, and it cannot be omitted. For instance, if $\tau_1 = \tau_2$ then $L_n(e_1 - e_2)$ could be very small: up to replacing (e_1, e_2) with $(e_1 + e_2, e_1 - e_2)$, this amounts to dropping the assumption that the linear forms L_n are not too small at the points e_j . Now this assumption is known to be essential, already in the classical case of Nesterenko's linear independence criterion (except for proving the linear independence of three numbers, see Theorem 2 of [12]). Actually, if $\tau_1 = \tau_2$ then $L_n(e_1 - e_2)$ could even vanish, so the possibility that $e_1 = e_2$ cannot be eliminated: even assertion (ii) may fail to hold.

Under the assumptions of Theorem 3, it can be useful to apply this result (in the qualitative form (i)) to the point $\lambda_1 e_1 + \dots + \lambda_k e_k$ with fixed $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ (which amounts to applying Nesterenko's original criterion, generalized [8] to allow oscillatory behaviours). This is how Theorem 2 is proved, and also the above-mentioned results of [17] and [10] (with $k = 2$). However, the quantitative conclusion (iii) of Theorem 3 itself contains all quantitative results that can be obtained in this way (because the latter involve smaller convex bodies \mathcal{C}), so that it implies the qualitative statement (i) for any linear combination $\lambda_1 e_1 + \dots + \lambda_k e_k$ using Minkowski's convex body theorem.

At last, we remark that the lower bound $k + \tau_1 + \dots + \tau_k$ in (i) is optimal (see [9] for a converse statement). It seems natural to imagine that (iii) is essentially optimal too: up to Q^ε , \mathcal{C} is defined as the set of all x such that the assumptions on L_n imply $|L_n(x)| < 1$, so that $L_n(x) = 0$ if $x \in \mathcal{C} \cap \mathbb{Z}^p$, where n is chosen such that $Q \approx Q_n$. When $p = 2$, $k = 1$, and $e_1 = (1, \xi)$, Theorem 3 (iii) yields an upper bound $\mu(\xi) \leq 1 + \frac{1}{\tau_1}$ on the irrationality exponent of ξ , and reduces essentially to Lemma 1 of [11]. A converse statement in this case is proved in [11] (Theorem 1).

The structure of this text is as follows. In §2 we give some corollaries of Theorem 3, in the setting of Padé approximation (§2.1) or in relation to the approximation of a fixed subspace by points of \mathbb{Z}^p (§2.2). We also study in §2.3 the rank of vectors in \mathbb{R}^k seen as a

vector space over the rationals; this allows us to prove the equivalence of both statements of (i) in Theorem 3, and the fact that Theorem 1 is a strict improvement on Ball-Rivoal's result (1.1) on zeta values. A linear independence criterion in the style of Siegel's is proved in §2.4.

Then we prove Theorem 3 in §3; the main new step is detailed in §3.4. At last, §4 is devoted to the applications to zeta values.

2 Consequences and related results

2.1 Connection with Padé approximation

To begin with, let us state Theorem 3 in terms of $C_1, \dots, C_p \in \mathbb{R}^k$.

Theorem 4. *Let $C_1, \dots, C_p \in \mathbb{R}^k$, with $k, p \geq 1$.*

Let $\tau_1, \dots, \tau_k, \omega_1, \dots, \omega_k, \varphi_1, \dots, \varphi_k$ and $(Q_n)_{n \geq 1}$ be as in Theorem 3.

For any $n \geq 1$, let $\ell_{1,n}, \dots, \ell_{p,n} \in \mathbb{Z}$ be such that, as $n \rightarrow \infty$:

$$\max_{1 \leq i \leq p} |\ell_{i,n}| \leq Q_n^{1+o(1)}$$

and

$$\ell_{1,n}C_1 + \dots + \ell_{p,n}C_p = \begin{pmatrix} \pm Q_n^{-\tau_1+o(1)}(\cos(n\omega_1 + \varphi_1) + o(1)) \\ \vdots \\ \pm Q_n^{-\tau_k+o(1)}(\cos(n\omega_k + \varphi_k) + o(1)) \end{pmatrix} \quad (2.1)$$

where the \pm signs can be independent from one another. Then:

(i) *We have*

$$\text{rk}_{\mathbb{Q}}(C_1, \dots, C_p) \geq k + \tau_1 + \dots + \tau_k,$$

where $\text{rk}_{\mathbb{Q}}(C_1, \dots, C_p)$ is the rank of the family of vectors C_1, \dots, C_p in \mathbb{R}^k , considered as a \mathbb{Q} -vector space.

(ii) *For any non-zero linear form $\chi : \mathbb{R}^k \rightarrow \mathbb{R}$ there exists $i \in \{1, \dots, p\}$ such that $\chi(C_i) \notin \mathbb{Q}$.*

(iii) *Let $\varepsilon > 0$, and Q be sufficiently large in terms of ε . Let $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, not all zero, be such that $|\lambda_j| \leq Q^{\tau_j}$ for any $j \in \{1, \dots, k\}$. Then denoting by χ the linear map $\mathbb{R}^k \rightarrow \mathbb{R}$ defined by $\chi(x_1, \dots, x_k) = \lambda_1 x_1 + \dots + \lambda_k x_k$, we have*

$$\text{dist}\left((\chi(C_1), \dots, \chi(C_p)), \mathbb{Z}^p \setminus \{(0, \dots, 0)\}\right) \geq Q^{-1-\varepsilon}$$

where $\text{dist}(y, \mathbb{Z}^p \setminus \{(0, \dots, 0)\})$ is the minimal distance of $y \in \mathbb{R}^p$ to a non-zero integer point.

This result is just a translation of Theorem 3. Indeed let us consider the matrix $M \in \text{Mat}_{k,p}(\mathbb{R})$ of which C_1, \dots, C_p are the columns. We denote by $e_1, \dots, e_k \in \mathbb{R}^p$ the rows of M . Then assumption (2.1) means that the linear form $L_n = \ell_{1,n}X_1 + \dots + \ell_{p,n}X_p$ on \mathbb{R}^p is small at the points e_1, \dots, e_k . It is not difficult to see that (ii) and (iii) in Theorem 4 are respectively equivalent to (ii) and (iii) in Theorem 3, because $(\chi(C_1), \dots, \chi(C_p)) = \lambda_1 e_1 + \dots + \lambda_k e_k$. We remark also that assuming $k \leq p-1$ in Theorem 3 is not necessary; it won't be used in the proof. Anyway this upper bound follows from (ii), so that it is actually a consequence of the other assumptions.

Let us focus now on an important special case of Theorem 4, related to Padé approximation: when C_1, \dots, C_k is the canonical basis of \mathbb{R}^k . This happens in all practical situations mentioned in this paper, including Theorem 1 and Eqns. (2.3), (2.4) and (2.5) in §2.4 below: indeed Padé approximation provides linear combinations of C_{k+1}, \dots, C_p which are very close to \mathbb{Z}^k . In this case, in (ii) the interesting point is when the linear form $\chi(x_1, \dots, x_k) = \lambda_1 x_1 + \dots + \lambda_k x_k$ has rational coefficients λ_j ; then we have $\chi(C_i) \notin \mathbb{Q}$ for some $i \in \{k+1, \dots, p\}$. An analogous remark holds for (iii); both are more easily stated as follows, in terms of e_1, \dots, e_k . We denote by $\|\cdot\|$ any fixed norm on \mathbb{R}^{p-k} .

Corollary 2. *Under the assumptions of Theorem 3, suppose that for any $j \in \{1, \dots, k\}$ we have $e_j = (0, \dots, 0, 1, 0, \dots, 0, e'_j)$ with $e'_j \in \mathbb{R}^{p-k}$, where the 1 is in j -th position.*

Then no non-trivial \mathbb{Q} -linear combination of e'_1, \dots, e'_k belongs to \mathbb{Q}^{p-k} . In addition, let $\varepsilon > 0$, and Q be sufficiently large in terms of ε . Let $\lambda_1, \dots, \lambda_k \in \mathbb{Z}$, not all zero, be such that $|\lambda_j| \leq Q^{\tau_j}$ for any $j \in \{1, \dots, k\}$. Then for any $S \in \mathbb{Z}^{p-k}$ we have

$$\|\lambda_1 e'_1 + \dots + \lambda_k e'_k - S\| \geq Q^{-1-\varepsilon}.$$

This corollary is a measure of linear independence of the vectors e'_1, \dots, e'_k and those of the canonical basis of \mathbb{Z}^{p-k} . It can be weakened by assuming $|\lambda_j| \leq Q^\tau$ for any $j \in \{1, \dots, k\}$, where $\tau = \min(\tau_1, \dots, \tau_k)$ (as in Theorem 5 below). Then a *measure of non-discreteness* (in the sense of [14]) is obtained for the lattice $\mathbb{Z}e'_1 + \dots + \mathbb{Z}e'_k + \mathbb{Z}^{p-k}$, which has rank p . In the examples (2.3), (2.4) and (2.5) considered in §2.4 below, the matrix with columns C_{k+1}, \dots, C_p is symmetric (with $p = 2k$), so that this lattice is exactly $\mathbb{Z}C_1 + \dots + \mathbb{Z}C_p$ (using the fact that C_1, \dots, C_k is the canonical basis of \mathbb{R}^k).

This case $k = p/2$ lies “in the middle” between $k = 1$, which corresponds to type I Padé approximation and Nesterenko's original criterion, and $k = p-1$, which corresponds to type II Padé approximation. In the latter case, Corollary 2 yields the following result by letting $\xi_j = -e'_j$.

Corollary 3. *Let $k \geq 1$, and $\xi_1, \dots, \xi_k \in \mathbb{R}$.*

Let $\tau_1, \dots, \tau_k > 0$ be pairwise distinct real numbers.

Let $(Q_n)_{n \geq 1}$ be an increasing sequence of positive integers, such that $Q_{n+1} = Q_n^{1+o(1)}$.

For any $n \geq 1$, let $\ell_{1,n}, \dots, \ell_{k,n}, \ell_{k+1,n} \in \mathbb{Z}$ be such that

$$\max_{1 \leq i \leq k+1} |\ell_{i,n}| \leq Q_n^{1+o(1)}$$

and

$$|\ell_{k+1,n}\xi_j - \ell_{j,n}| = Q_n^{-\tau_j+o(1)} \text{ for any } j \in \{1, \dots, k\}.$$

Then:

- (i) The numbers $1, \xi_1, \dots, \xi_k$ are \mathbb{Q} -linearly independent.
- (ii) Let $\varepsilon > 0$, and Q be sufficiently large (in terms of ε). Then for any $(a_0, a_1, \dots, a_k) \in \mathbb{Z}^{k+1} \setminus \{(0, \dots, 0)\}$ with $|a_j| \leq Q^{\tau_j}$ for any $j \in \{1, \dots, k\}$, we have:

$$|a_0 + a_1\xi_1 + \dots + a_k\xi_k| \geq Q^{-1-\varepsilon}.$$

We have not found this statement in the literature; see however [7] (p. 98), [15] (Lemma 2.1) or [16] (Lemma 6.1) for related results, which are maybe closer to Siegel's criterion than to Nesterenko's.

A direct proof of Corollary 3 can be derived from the equality

$$|\ell_{k+1,n}| \cdot |a_0 + a_1\xi_1 + \dots + a_k\xi_k| = \left| \sum_{j=1}^k a_j(\ell_{k+1,n}\xi_j - \ell_{j,n}) + \ell_{k+1,n}a_0 + \sum_{j=1}^k a_j\ell_{j,n} \right|$$

by arguing as in §3.2 for (i), and by taking n maximal such that $Q_n \cdot |a_0 + a_1\xi_1 + \dots + a_k\xi_k| < \frac{1}{2}$ for (ii).

2.2 Upper bound on a Diophantine exponent

Given a subspace F of \mathbb{R}^p , and a non-zero point $P \in \mathbb{R}^p$, we denote by $\text{Dist}(P, F)$ the projective distance of P to F , seen in $\mathbb{P}^p(\mathbb{R})$. Several definitions may be given, all of them equivalent up to multiplicative constants (see for instance [25]); we choose $\text{Dist}(P, F) = \frac{\|u\|}{\|P\|}$ where u is the orthogonal projection of P on F^\perp (that is, P can be written as $u + f$ with $u \in F^\perp$ and $f \in F$), and $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^p .

The following result is a consequence of Theorem 3.

Theorem 5. *Under the assumptions of Theorem 3, let $\tau = \min(\tau_1, \dots, \tau_k)$ and $F = \text{Span}_{\mathbb{R}}(e_1, \dots, e_k)$. Then for any $\varepsilon > 0$ and any $P \in \mathbb{Z}^p \setminus \{(0, \dots, 0)\}$ we have:*

$$\text{Dist}(P, F) \geq \|P\|^{-1-\frac{1}{\tau}-\varepsilon}$$

provided $\|P\|$ is sufficiently large in terms of ε .

It is important to notice that Theorem 5 is *not* optimal, since it involves only $\min(\tau_1, \dots, \tau_k)$. It is specially interesting when τ_1, \dots, τ_k are close to one another.

The interest of Theorem 5 is that it can be written as an upper bound on a Diophantine exponent which measures the approximation of F by points of \mathbb{Z}^p (see [25], [19], [5]).

Proof of Theorem 5: Using assertion (ii) of Theorem 3, we see that (e_1, \dots, e_k) is a basis of F . Since F is finite-dimensional, all norms on F are equivalent: there exists $\kappa > 0$ such that, for any $f = \lambda_1 e_1 + \dots + \lambda_k e_k \in F$ (with $\lambda_j \in \mathbb{R}$), we have $\max |\lambda_j| \leq \kappa \|f\|$.

Let $\varepsilon > 0$. Let Q_0 be such that assertion (iii) of Theorem 3 holds for any $Q \geq Q_0$; we assume that $\|P\| \geq Q_0^\tau / \kappa$. Letting $Q = (\kappa \|P\|)^{1/\tau}$ we have $Q \geq Q_0$. Since $P \in \mathbb{Z}^p \setminus \{(0, \dots, 0)\}$, P does not belong to the set \mathcal{C} defined in assertion (iii). Now writing $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u$ with $\lambda_j \in \mathbb{R}$ and $u \in F^\perp$, we have

$$\max_{1 \leq j \leq k} |\lambda_j| \leq \kappa \|\lambda_1 e_1 + \dots + \lambda_k e_k\| \leq \kappa \|P\| = Q^\tau$$

so that $\|u\| > Q^{-1-\varepsilon}$. Using the definition of Q and that of $\text{Dist}(P, F)$, this concludes the proof of Theorem 5.

2.3 Rational rank of vectors

In this section, we give some details about the conclusions of our criterion and prove that Theorem 1 is a strict improvement on Ball-Rivoal's result [2] on zeta values.

In Nesterenko's linear independence criterion, a lower bound is derived for the dimension of the \mathbb{Q} -subspace of \mathbb{R} spanned by $\xi_0, \dots, \xi_r \in \mathbb{R}$, that is, for the \mathbb{Q} -rank of ξ_0, \dots, ξ_r in \mathbb{R} considered as a vector space over \mathbb{Q} . This rank is equal to the dimension of the smallest subspace of \mathbb{R}^{r+1} , defined over the rationals, which contains the point (ξ_0, \dots, ξ_r) . We generalize in Lemma 1 below this equality to our setting.

Recall that a subspace F of \mathbb{R}^p is said to be *defined over* \mathbb{Q} if it is the zero locus of a family of linear forms with rational coefficients. This is equivalent to the existence of a basis (or a generating family) of F , as a vector space over \mathbb{R} , consisting in vectors of \mathbb{Q}^p (see for instance §8 of [4]). Since the intersection of a family of subspaces of \mathbb{R}^p defined over \mathbb{Q} is again defined over \mathbb{Q} , there exists for any subset $S \subset \mathbb{R}^p$ a minimal subspace of \mathbb{R}^p , defined over \mathbb{Q} , which contains S : this is the intersection of all subspaces of \mathbb{R}^p , defined over \mathbb{Q} , which contain S .

Let M be a matrix with $k \geq 1$ rows, $p \geq 1$ columns, and real entries. Letting $e_1, \dots, e_k \in \mathbb{R}^p$ denote the rows of M , we can consider as above the smallest subspace of \mathbb{R}^p , defined over \mathbb{Q} , which contains e_1, \dots, e_k . On the other hand, we denote by $C_1, \dots, C_p \in \mathbb{R}^k$ the columns of M and consider \mathbb{R}^k as an infinite-dimensional vector space over \mathbb{Q} . Then $\text{Span}_{\mathbb{Q}}(C_1, \dots, C_p)$ is the smallest \mathbb{Q} -vector subspace of \mathbb{R}^k containing C_1, \dots, C_p ; it consists in all linear combinations $r_1 C_1 + \dots + r_p C_p$ with $r_1, \dots, r_p \in \mathbb{Q}$. Its dimension (as a \mathbb{Q} -vector space) is the rank (over \mathbb{Q}) of C_1, \dots, C_p , denoted by $\text{rk}_{\mathbb{Q}}(C_1, \dots, C_p)$.

Lemma 1. *Let $M \in \text{Mat}_{k,p}(\mathbb{R})$ with $k, p \geq 1$. Denote by $e_1, \dots, e_k \in \mathbb{R}^p$ the rows of M , and by $C_1, \dots, C_p \in \mathbb{R}^k$ its columns. Then $\text{rk}_{\mathbb{Q}}(C_1, \dots, C_p)$ is the dimension of the smallest subspace of \mathbb{R}^p , defined over \mathbb{Q} , which contains e_1, \dots, e_k .*

When $k = 1$, this lemma means that the \mathbb{Q} -rank of ξ_0, \dots, ξ_r is equal to the dimension of the smallest subspace of \mathbb{R}^{r+1} , defined over the rationals, which contains the point (ξ_0, \dots, ξ_r) .

Proof of Lemma 1: Let $G = (\text{Span}_{\mathbb{R}}(e_1, \dots, e_k))^\perp$, where \mathbb{R}^p is equipped with the usual scalar product. Let F denote the minimal subspace of \mathbb{R}^p , defined over \mathbb{Q} , which contains e_1, \dots, e_k . Then F^\perp is the maximal subspace of \mathbb{R}^p , defined over \mathbb{Q} , which is contained in $G = \{e_1, \dots, e_k\}^\perp$. Therefore $F^\perp = \text{Span}_{\mathbb{R}}(G \cap \mathbb{Q}^p) = (G \cap \mathbb{Q}^p) \otimes_{\mathbb{Q}} \mathbb{R}$: any basis of the \mathbb{Q} -vector space $G \cap \mathbb{Q}^p$ is an \mathbb{R} -basis of F^\perp . Since $G \cap \mathbb{Q}^p = \ker \psi$ where $\psi : \mathbb{Q}^p \rightarrow \mathbb{R}^k$ is defined by $\psi(r_1, \dots, r_p) = r_1 C_1 + \dots + r_p C_p$, we have:

$$\dim_{\mathbb{R}} F = p - \dim_{\mathbb{R}} F^\perp = p - \dim_{\mathbb{Q}}(G \cap \mathbb{Q}^p) = \text{rk}_{\mathbb{Q}} \psi = \text{rk}_{\mathbb{Q}}(C_1, \dots, C_p).$$

This concludes the proof of Lemma 1.

Let us proceed now to prove that Theorem 1 is a strict improvement on Ball-Rivoal's result [2] on zeta values.

Let $a \geq 3$ be an odd integer, and $\xi_3, \xi_5, \dots, \xi_{a+2}$ be real numbers. We let $\underline{\xi} = (\xi_3, \xi_5, \dots, \xi_{a+2})$,

$$n(\underline{\xi}) = \dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \xi_3, \xi_5, \dots, \xi_a),$$

and $N(\underline{\xi})$ denote the dimension of the minimal subspace of $\mathbb{R}^{(a+3)/2}$, defined over \mathbb{Q} , which contains both points $(1, 0, \xi_3, \xi_5, \dots, \xi_a)$ and $(0, 1, \binom{4}{2}\xi_5, \binom{6}{2}\xi_7, \dots, \binom{a+1}{2}\xi_{a+2})$. As Lemma 1 shows, $N(\underline{\xi})$ is the \mathbb{Q} -rank of the family of vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \xi_3 \\ \binom{4}{2}\xi_5 \end{pmatrix}, \begin{pmatrix} \xi_5 \\ \binom{6}{2}\xi_7 \end{pmatrix}, \dots, \begin{pmatrix} \xi_a \\ \binom{a+1}{2}\xi_{a+2} \end{pmatrix}.$$

When $\underline{\xi} = (\zeta(3), \zeta(5), \dots, \zeta(a+2))$ and $a \rightarrow \infty$, Ball-Rivoal's result reads $n(\underline{\xi}) \geq \frac{\log a}{1+\log 2}(1+o(1))$ whereas Theorem 1 states that $N(\underline{\xi}) \geq \frac{2 \log a}{1+\log 2}(1+o(1))$. The latter is a strict improvement on the former, as the following lemma shows.

Lemma 2. *With the previous notation:*

- (i) *For any $\underline{\xi}$ we have $n(\underline{\xi}) + 1 \leq N(\underline{\xi}) \leq 2n(\underline{\xi}) + 1$.*
- (ii) *For any positive integers n and N with $N \leq \frac{a+3}{2}$ and $n+1 \leq N \leq 2n+1$ there exists $\underline{\xi}$ such that $n(\underline{\xi}) = n$ and $N(\underline{\xi}) = N$.*

Proof of Lemma 2:

(i) Let F_1 (resp. F_2 , resp. F) be the minimal subspace of $\mathbb{R}^{(a+3)/2}$, defined over \mathbb{Q} , which contains the point $e_1 = (1, 0, \xi_3, \xi_5, \dots, \xi_a)$ (resp. $e_2 = (0, 1, \binom{4}{2}\xi_5, \binom{6}{2}\xi_7, \dots, \binom{a+1}{2}\xi_{a+2})$, resp. both e_1 and e_2). We have $\dim F_1 = n(\underline{\xi})$ and $\dim F_2 = \dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \xi_5, \xi_7, \dots, \xi_{a+2}) \leq n(\underline{\xi}) + 1$. Since $F_1 + F_2$ is a subspace of $\mathbb{R}^{(a+3)/2}$ defined over \mathbb{Q} which contains both e_1 and e_2 , we have $N(\underline{\xi}) \leq \dim(F_1 + F_2) \leq 2n(\underline{\xi}) + 1$. On the other hand, denoting by H the hyperplane of $\mathbb{R}^{(a+3)/2}$ defined by the vanishing of the second coordinate, we have $F_1 \subset F \cap H \subsetneq F$ (by minimality of F_1 and because $e_2 \notin H$), so that $n(\underline{\xi}) \leq N(\underline{\xi}) - 1$.

(ii) Let $i = 2n - 1$ and $j = 2N - 3$, so that $1 \leq i \leq j \leq a$ and $j \leq 2i + 1$; obviously i and j are positive odd integers. If $i \geq 3$ we choose $\xi_3, \xi_5, \dots, \xi_i \in \mathbb{R}$ such that $1, \xi_3, \xi_5, \dots, \xi_i$

are \mathbb{Q} -linearly independent. If $j \geq i + 2$ we let $\xi_{i+2} = 1$, $\xi_{i+4} = \xi_3, \dots, \xi_j = \xi_{j-i-1}$ (where $\xi_1 = 1$). At last, we let $\xi_{j+2} = \xi_{j+4} = \dots = \xi_{a+2} = 0$. Then we have $n(\underline{\xi}) = n$ by construction.

For any odd integer $k \in \{3, 5, \dots, a\}$, let $C_k = \begin{pmatrix} \xi_k \\ \binom{k+1}{2} \xi_{k+2} \end{pmatrix} \in \mathbb{R}^2$; let also $C_{-1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $C_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $N(\underline{\xi}) = \text{rk}_{\mathbb{Q}}(C_{-1}, C_1, C_3, C_5, \dots, C_a)$ using Lemma 1. Since $C_{j+2} = C_{j+4} = \dots = C_a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we have $N(\underline{\xi}) = \text{rk}_{\mathbb{Q}}(C_{-1}, C_1, \dots, C_j)$. Let us prove that these $\frac{j+3}{2} = N$ vectors are \mathbb{Q} -linearly independent; this will conclude the proof that $N(\underline{\xi}) = N$.

Let $\lambda_{-1}, \lambda_1, \lambda_3, \dots, \lambda_j \in \mathbb{Q}$ be such that $\lambda_{-1}C_{-1} + \lambda_1C_1 + \lambda_3C_3 + \dots + \lambda_jC_j = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This means

$$\begin{cases} \lambda_{-1} + \lambda_{i+2} + (\lambda_3 + \lambda_{i+4})\xi_3 + (\lambda_5 + \lambda_{i+6})\xi_5 + \dots + (\lambda_{j-i-1} + \lambda_j)\xi_{j-i-1} \\ \quad + \lambda_{j-i+1}\xi_{j-i+1} + \lambda_{j-i+3}\xi_{j-i+3} + \dots + \lambda_i\xi_i = 0 \\ \lambda_1 + \binom{i+1}{2}\lambda_i + \binom{i+3}{2}\lambda_{i+2}\xi_3 + [\binom{4}{2}\lambda_3 + \binom{i+5}{2}\lambda_{i+4}]\xi_5 + [\binom{6}{2}\lambda_5 + \binom{i+7}{2}\lambda_{i+6}]\xi_7 \\ \quad + \dots + [\binom{j-i-2}{2}\lambda_{j-i-3} + \binom{j-1}{2}\lambda_{j-2}]\xi_{j-i-1} \\ \quad + \binom{j-i}{2}\lambda_{j-i-1}\xi_{j-i+1} + \binom{j-i+2}{2}\lambda_{j-i+1}\xi_{j-i+3} + \dots + \binom{i-1}{2}\lambda_{i-2}\xi_i = 0. \end{cases}$$

Since $1, \xi_3, \xi_5, \dots, \xi_i$ are \mathbb{Q} -linearly independent, this implies

$$\begin{cases} \lambda_{-1} + \lambda_{i+2} = \lambda_3 + \lambda_{i+4} = \lambda_5 + \lambda_{i+6} = \dots = \lambda_{j-i-1} + \lambda_j = 0 \\ \lambda_1 + \binom{i+1}{2}\lambda_i = \binom{i+3}{2}\lambda_{i+2} = \binom{4}{2}\lambda_3 + \binom{i+5}{2}\lambda_{i+4} = \binom{6}{2}\lambda_5 + \binom{i+7}{2}\lambda_{i+6} \\ \quad = \dots = \binom{j-i-2}{2}\lambda_{j-i-3} + \binom{j-1}{2}\lambda_{j-2} = 0 \\ \lambda_{j-i-1} = \lambda_{j-i+1} = \dots = \lambda_i = 0. \end{cases}$$

Since $\binom{t}{2} \neq \binom{t+i+1}{2}$ for any $t \in \{4, 6, \dots, j-i-2\}$, this is easily seen to imply $\lambda_{-1} = \lambda_1 = \lambda_3 = \dots = \lambda_j = 0$. This concludes the proof of Lemma 2.

2.4 Connection with a Siegel-type criterion

The following result is analogous to Theorem 3, but its proof is much easier. It relies on Siegel's ideas for linear independence (see for instance [7], p. 81–82 and 215–216, or [20], Proposition 4.1).

Proposition 1. *Let $1 \leq k \leq p-1$, and $e_1, \dots, e_k \in \mathbb{R}^p$ be linearly independent vectors.*

Let $(Q_n)_{n \geq 1}$ be an increasing sequence of positive integers, and for any $n \geq 1$, let $L_n^{(t)} = \ell_{1,n}^{(t)}X_1 + \dots + \ell_{p,n}^{(t)}X_p$ be p linearly independent linear forms on \mathbb{R}^p (for $1 \leq t \leq p$), with integer coefficients $\ell_{i,n}^{(t)}$ such that, as $n \rightarrow \infty$:

$$|L_n^{(t)}(e_j)| \leq Q_n^{-\tau_j + o(1)} \text{ for any } j \in \{1, \dots, k\} \text{ and any } t \in \{1, \dots, p\},$$

where $\tau_1, \dots, \tau_k > 0$ are real numbers, and

$$\max_{\substack{1 \leq i \leq p \\ 1 \leq t \leq p}} |\ell_{i,n}^{(t)}| \leq Q_n^{1+o(1)}.$$

Then:

(a) Conclusions (i) and (ii) of Theorem 3 hold.

(b) Let $\varepsilon > 0$, and n be sufficiently large (in terms of ε). Let \mathcal{C}_n denote the set of all vectors that can be written as $\lambda_1 e_1 + \dots + \lambda_k e_k + u$ with:

$$\begin{cases} \lambda_1, \dots, \lambda_k \in \mathbb{R} \text{ such that } |\lambda_j| \leq Q_n^{\tau_j - \varepsilon} \text{ for any } j \in \{1, \dots, k\} \\ u \in (\text{Span}_{\mathbb{R}}(e_1, \dots, e_k))^{\perp} \text{ such that } \|u\| \leq Q_n^{-1 - \varepsilon} \end{cases}$$

Then $\mathcal{C}_n \cap \mathbb{Z}^p = \{(0, \dots, 0)\}$.

The main difference with Theorem 3 is that we require here p linearly independent linear forms for any n . This makes the proof much easier, and enables one to get rid of several important assumptions of Theorem 3 (namely $Q_{n+1} = Q_n^{1+o(1)}$, τ_1, \dots, τ_k pairwise distinct, and $|L_n(e_j)|$ not too small).

If $Q_{n+1} = Q_n^{1+o(1)}$ in Proposition 1 then in (b) we may replace Q_n with any Q , by letting n be such that $Q_n \leq Q < Q_{n+1}$.

Proof of Proposition 1: To prove conclusion (i) of Theorem 3, let F be a subspace of \mathbb{R}^p defined over \mathbb{Q} , of dimension d , which contains e_1, \dots, e_k . Let n be sufficiently large. Up to reordering $L_n^{(1)}, \dots, L_n^{(p)}$, we may assume the restrictions of $L_n^{(1)}, \dots, L_n^{(d)}$ to F to be linearly independent linear forms on F . Denoting by (u_1, \dots, u_d) a basis of F consisting in vectors of \mathbb{Z}^p , the matrix $[L_n^{(t)}(u_j)]_{1 \leq t, j \leq d}$ has a non-zero integer determinant. By making suitable linear combinations of the columns, the values $L_n^{(t)}(e_1), \dots, L_n^{(t)}(e_k)$ appear and lead to the upper bound $Q_n^{d-k-\tau_1-\dots-\tau_k+o(1)}$ on the absolute value of this determinant. This concludes the proof of (i) of Theorem 3.

To prove part (b) of Proposition 1 (which implies conclusion (ii) of Theorem 3), we let $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u \in \mathcal{C}_n \cap \mathbb{Z}^p$ be non-zero; then $L_n^{(t)}(P) \neq 0$ for some t , but $L_n^{(t)}(P) \in \mathbb{Z}$ and $|L_n^{(t)}(P)| < 1$. This concludes the proof of Proposition 1.

This criterion is used (in its qualitative part (a), and in the special case of $k = 1$ point) by Nikishin [22] to prove that $1, \text{Li}_1(z), \text{Li}_2(z), \dots, \text{Li}_a(z)$ are \mathbb{Q} -linearly independent provided that $z = p/q < 0$ where p and q are integers and q is sufficiently large in terms of $|p|$ and a ; here $\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ is the s -th polylogarithm, for $s \geq 1$. Marcovecchio has given [20] a proof of Rivoal's result [24]

$$\dim_{\mathbb{Q}} \text{Span}_{\mathbb{Q}}(1, \text{Li}_1(z), \text{Li}_2(z), \dots, \text{Li}_a(z)) \geq \frac{\log a}{1 + \log 2} (1 + o(1)) \quad (2.2)$$

as $a \rightarrow \infty$, for any $z \in \mathbb{Q}$ such that $|z| < 1$, based on Siegel's ideas (whereas Rivoal used Nesterenko's criterion). However no such proof is known for Ball-Rivoal's theorem (1.1).

Using essentially Proposition 1 with $k = 2$, Gutnik proved [13] that the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \log 2 \\ \zeta(2) \end{pmatrix}, \begin{pmatrix} \zeta(2) \\ -3\zeta(3) \end{pmatrix} \quad (2.3)$$

are \mathbb{Q} -linearly independent in \mathbb{R}^2 (so that, for any $r \in \mathbb{Q}^*$, at least one number among $\zeta(2) - 2r \log 2$ and $3\zeta(3) - r\zeta(2)$ is irrational). More recently he obtained also [14] the \mathbb{Q} -linear independence of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2\zeta(3) \\ 3\zeta(4) \end{pmatrix}, \begin{pmatrix} 3\zeta(4) \\ 6\zeta(5) \end{pmatrix}. \quad (2.4)$$

Such results of linear independence are known only for at most 4 vectors (analogously to the irrationality of odd zeta values, known only for $\zeta(3)$). To obtain the linear independence of more than 4 vectors, it is necessary (with the present methods) to consider polylogarithms at points z close to 0, as in Nikishin's result. In this direction, T. Hessami-Pilehrood has proved [18] that if q is greater than some explicit function of k then the following $2k$ vectors are \mathbb{Q} -linearly independent in \mathbb{R}^k :

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad (2.5)$$

$$\begin{pmatrix} \text{Li}_1(\frac{-1}{q}) \\ \text{Li}_2(\frac{-1}{q}) \\ \vdots \\ \text{Li}_k(\frac{-1}{q}) \end{pmatrix}, \dots, \begin{pmatrix} \binom{j-1}{j-1} \text{Li}_j(\frac{-1}{q}) \\ \binom{j}{j-1} \text{Li}_{j+1}(\frac{-1}{q}) \\ \vdots \\ \binom{j+k-2}{j-1} \text{Li}_{j+k-1}(\frac{-1}{q}) \end{pmatrix}, \dots, \begin{pmatrix} \binom{k-1}{k-1} \text{Li}_k(\frac{-1}{q}) \\ \binom{k}{k-1} \text{Li}_{k+1}(\frac{-1}{q}) \\ \vdots \\ \binom{2k-2}{k-1} \text{Li}_{2k-1}(\frac{-1}{q}) \end{pmatrix}.$$

The same result holds with $1/q$ instead of $-1/q$; see also Gutnik's preprints cited in [18].

The proofs of (2.3), (2.4) and (2.5) rely on the same principle: in addition to a “derivation procedure” as in the definition of S''_n (see the introduction), they involve constructing several sequences of linearly independent small linear forms, as in Nikishin's proof, and using essentially Proposition 1. All of them lead to the \mathbb{Q} -linear independence of the whole set of p vectors under consideration (with strict restrictions: few vectors or a point z close to 0), because Proposition 1 is applied with $k + \tau_1 + \dots + \tau_k > p - 1$ so that conclusion (a) yields $\text{rk}_{\mathbb{Q}}(C_1, \dots, C_p) \geq p$. However the proofs can probably be modified to provide lower bounds such as Eq. (2.2).

3 Proof of the criterion

This section is devoted to proving Theorem 3. Reindexing e_1, \dots, e_k is necessary, we assume that $\tau_1 > \dots > \tau_k > 0$. This will be used in §§3.2 and 3.5.

3.1 Reduction to the non-oscillatory case

In this subsection, we deduce the general case of Theorem 3 from the special case where $\omega_1 = \dots = \omega_k = 0$; notice that in this case we have $\phi_j \not\equiv \frac{\pi}{2} \pmod{\pi}$ for any $j \in \{1, \dots, k\}$,

so that Eq. (1.2) reads $|L_n(e_j)| = Q_n^{-\tau_j+o(1)}$. This special case will be proved in the following subsections, under the assumption that $Q_{n+1} = Q_n^{1+o(1)}$ (which is weaker than the assumption $Q_{n+1} = Q_n^{1+O(1/n)}$ we make when $\omega_1, \dots, \omega_k$ may be non-zero).

Let $\omega_1, \dots, \omega_k, \varphi_1, \dots, \varphi_k$, and (Q_n) be as in Theorem 3, with $Q_{n+1} = Q_n^{1+O(1/n)}$. Since there are infinitely many integers n such that, for any $j \in \{1, \dots, k\}$, $n\omega_j + \varphi_j \not\equiv \frac{\pi}{2} \pmod{\pi}$, Proposition 1 of [8] provides $\varepsilon, \lambda > 0$ and an increasing function $\psi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\lim_{n \rightarrow \infty} \frac{\psi(n)}{n} = \lambda$ and, for any n and any $j \in \{1, \dots, k\}$, $|\cos(\psi(n)\omega_j + \varphi_j)| \geq \varepsilon$. Let $L'_n = L_{\psi(n)}$ and $Q'_n = Q_{\psi(n)}$ for any $n \geq 1$. Then we have $|L'_n(e_j)| = Q_n^{-\tau_j+o(1)}$ because $|\cos(\psi(n)\omega_j + \varphi_j)| = Q_n^{o(1)}$. Let us check that $Q'_{n+1} = Q_n^{1+o(1)}$; then the special case of Theorem 3 will apply to the sequences $(L'_n)_{n \geq 1}$ and $(Q'_n)_{n \geq 1}$, with the same $e_1, \dots, e_k, \tau_1, \dots, \tau_k$: this will conclude the proof.

Since $Q_{n+1} = Q_n^{1+O(1/n)}$ there exists $M > 0$ such that, for any $n \geq 1$, $Q_{n+1} \leq Q_n^{1+M/n}$; this implies

$$Q_{n+\ell} \leq Q_n^{(1+M/n)^\ell}$$

for any $\ell \geq 0$. Letting $\delta_n = \psi(n+1) - \psi(n) \geq 1$, we have:

$$Q'_{n+1} = Q_{\psi(n)+\delta_n} \leq Q_{\psi(n)}^{(1+M/\psi(n))^{\delta_n}} \leq Q_{\psi(n)}^{\exp(M\delta_n/\psi(n))} = Q_n^{1+o(1)}$$

by writing the exponent δ_n as $\frac{\psi(n)}{M} \frac{M\delta_n}{\psi(n)}$ with $(1 + M/\psi(n))^{\psi(n)/M} \leq e$, and using the fact that $\delta_n = o(n)$ since $\psi(n) = \lambda n + o(n)$. This concludes the reduction to the case where $\omega_1 = \dots = \omega_k = 0$ and $Q_{n+1} = Q_n^{1+o(1)}$.

3.2 Proof of (ii)

Let us start with the easiest part of Theorem 3, namely (ii).

We shall prove simultaneously that e_1, \dots, e_k are linearly independent in \mathbb{R}^p , and that $F \cap \mathbb{Q}^p = \{(0, \dots, 0)\}$ where $F = \text{Span}_{\mathbb{R}}(e_1, \dots, e_k)$. With this aim in mind, we assume (by contradiction) that there exist real numbers $\lambda_1, \dots, \lambda_k$, not all zero, such that $\sum_{j=1}^k \lambda_j e_j \in \mathbb{Q}^p$; multiplying all λ_j by a common denominator of the coordinates, we may assume $\sum_{j=1}^k \lambda_j e_j \in \mathbb{Z}^p$. Then $\kappa_n = L_n(\sum_{j=1}^k \lambda_j e_j) = \sum_{j=1}^k \lambda_j L_n(e_j)$ is an integer for any $n \geq 1$. Now if n is sufficiently large then $|\kappa_n| \leq \sum_{j=1}^k |\lambda_j| |L_n(e_j)| < 1$, so that $\kappa_n = 0$. Let j_0 denote the largest integer j such that $\lambda_j \neq 0$. Then for any n sufficiently large, the fact that $\kappa_n = 0$ implies $|\lambda_{j_0} L_n(e_{j_0})| = |\sum_{j=1}^{j_0-1} \lambda_j L_n(e_j)|$ so that

$$|\lambda_{j_0}| \leq \sum_{j=1}^{j_0-1} |\lambda_j| \frac{|L_n(e_j)|}{|L_n(e_{j_0})|} \leq \sum_{j=1}^{j_0-1} |\lambda_j| Q_n^{\tau_{j_0} - \tau_j + o(1)}$$

as $n \rightarrow \infty$. Now the right handside tends to 0 as $n \rightarrow \infty$ because we have assumed that $\tau_1 > \dots > \tau_k$, so that $\lambda_{j_0} = 0$: this contradicts the definition of λ_{j_0} .

Therefore such real numbers $\lambda_1, \dots, \lambda_k$ cannot exist, and this concludes the proof of (ii).

3.3 Proof that (ii) and (iii) imply (i)

Before proceeding in §§3.4 and 3.5 to the proof of (iii), which is the main part, we deduce (i) from (ii) and (iii). Recall that the second statement of (i) is equivalent to the first one (which we shall prove now) thanks to Lemma 1 proved in §2.3.

Let F be a subspace of \mathbb{R}^p , defined over \mathbb{Q} , which contains e_1, \dots, e_k . Letting $s = \dim F$, we have $s > k$ using (ii). Assertion (iii) yields, for any $\varepsilon > 0$ and any Q sufficiently large (in terms of ε), a subset \mathcal{C} of \mathbb{R}^s with no integer point other than $(0, \dots, 0)$. Now $\mathcal{C} \cap F$ is a convex body, compact and symmetric with respect to the origin, in the Euclidean space F . On the other hand, $\mathbb{Z}^p \cap F$ is a lattice in F because F is defined over \mathbb{Q} . Therefore Minkowski's convex body theorem (see for instance Chapter III of [6]) implies that $\mathcal{C} \cap F$ has volume less than $2^s \det(\mathbb{Z}^p \cap F)$. Now this volume is equal to $Q^{\tau_1 + \dots + \tau_k - (1+\varepsilon)(s-k)}$, up to a multiplicative constant which depends only on s, k, e_1, \dots, e_k . Since Q can be chosen arbitrarily large, we have $\tau_1 + \dots + \tau_k - (1+\varepsilon)(s-k) \leq 0$. Now ε can be any positive real number, so that $\tau_1 + \dots + \tau_k \leq s - k$, thereby concluding the proof of (i).

3.4 The main step

We state and prove in this section the main tool in the proof of Theorem 3, namely Proposition 2.

Recall that Nesterenko's linear independence criterion is much easier to prove if the linear forms $L_n, L_{n+1}, \dots, L_{n+p-1}$ are linearly independent (see §2.3 of [12] or the references to Siegel's criterion in the introduction). Of course this is not always the case, but Proposition 2 is a step in this direction. Actually letting $F = \text{Span}_{\mathbb{R}}(e_1, \dots, e_k)$, we consider the restrictions $L_n|_F$ of the linear forms to F ; recall that $\dim F = k$ thanks to (ii) proved in §3.2. It is not true in general that $L_n|_F, L_{n+1}|_F, \dots, L_{n+k-1}|_F$ are linearly independent linear forms on F : for instance, the equality $L_n = L_{n+1}$ might hold for any even integer n (because of the error terms $o(1)$ in the assumptions of Theorem 3). To make this statement correct, we introduce a function $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$ such that $\varphi(n) \geq n + 1$ for any $n \geq 1$: the integer $\varphi(n)$ plays the role of $n + 1$, that is: applying φ corresponds to "taking the next integer". The idea is that $\varphi(n)$ will be large enough (in comparison to n) to avoid obvious counter-examples as above coming from error terms. In the situation of Theorem 3, $\varphi(n)$ will be defined by the property $Q_{\varphi(n)-1} \leq Q_n^{1+\varepsilon_1} < Q_{\varphi(n)}$ (where ε_1 is a small positive real number); in this way, the error terms $o(1)$ in the assumptions of Theorem 3 will not be a problem any more.

With this definition, we are able to prove that for any n sufficiently large, the linear forms $L_n|_F, L_{\varphi(n)}|_F, L_{\varphi^2(n)}|_F, \dots, L_{\varphi^{k-1}(n)}|_F$ on F are linearly independent (where $\varphi^i = \varphi \circ \dots \circ \varphi$), so that they make up a basis of the dual vector space F^* . In the proof of Theorem 3 we shall need the following quantitative version of this property: in writing the linear form e_j^* (defined by $e_j^*(\lambda_1 e_1 + \dots + \lambda_k e_k) = \lambda_j$) as a linear combination of $\frac{1}{L_n(e_j)} L_n|_F, \frac{1}{L_{\varphi(n)}(e_j)} L_{\varphi(n)}|_F, \dots, \frac{1}{L_{\varphi^{k-1}(n)}(e_j)} L_{\varphi^{k-1}(n)}|_F$, the coefficients that appear are bounded independently from n (actually they are between -3 and 3).

We state and prove Proposition 2 in a general setting, with assumptions on φ that

make the proof work, because this result might be of independent interest. Its proof relies on estimating a $k \times k$ determinant.

Proposition 2. *Let $p > k > 0$, $e_1, \dots, e_k \in \mathbb{R}^p$ and $(L_n)_{n \geq 1}$ be a sequence of linear forms on \mathbb{R}^p . For any $n \geq 1$ and any $j \in \{1, \dots, k\}$, let $\varepsilon_{j,n} = |L_n(e_j)|$. Let $n_0 \geq 1$ be such that $\varepsilon_{j,n} \neq 0$ for any j and any $n \geq n_0$. Let also $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$ be such that*

$$\forall n \geq 1 \quad \varphi(n) \geq n + 1 \quad (3.1)$$

and

$$\forall i \in \{1, \dots, k-1\}, \forall n \geq n_0, \forall n' \geq \varphi(n) \quad \frac{\varepsilon_{i,n'}}{\varepsilon_{i,n}} \leq \frac{1}{(k+1)!} \frac{\varepsilon_{i+1,n'}}{\varepsilon_{i+1,n}}. \quad (3.2)$$

Let $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ and $M = \sum_{j=1}^k \lambda_j e_j$. Then for any $n \geq n_0$ and any $j \in \{1, \dots, k\}$ we have:

$$|\lambda_j| \leq \left(1 + \frac{1}{k} + \frac{1}{k^2}\right) \sum_{i=1}^k \frac{|L_{\varphi^{i-1}(n)}(M)|}{|L_{\varphi^{i-1}(n)}(e_j)|}.$$

Taking $M = 0$ implies that, under the assumptions of this proposition, e_1, \dots, e_k are linearly independent ; in our case this was proved already (in a simpler way) in §3.2.

Proposition 2 is optimal up to the value of the constant $1 + \frac{1}{k} + \frac{1}{k^2}$: it would be false with a constant less than $1/k$ instead (this is immediately seen by taking $M = e_j$). We did not try to improve on the constant $1 + \frac{1}{k} + \frac{1}{k^2}$, but anyway it could easily be made smaller by replacing $\frac{1}{(k+1)!}$ in (3.2) with a smaller constant.

Assumption (3.1) has been explained before the statement of Proposition 2. Assumption (3.2) means that $\frac{\varepsilon_{i,n}}{\varepsilon_{i+1,n}} \rightarrow 0$ as $n \rightarrow \infty$, and φ provides a control upon the rate of decrease of this sequence. In the proof of Theorem 3, this assumption will be checked in §3.5 using the assumption $\tau_i > \tau_{i+1}$.

In the proof of Proposition 2 we shall use the following lemma.

Lemma 3. *Under the assumptions of Proposition 2, for any $\sigma \in \mathfrak{S}_k$ and any $n \geq n_0$ we have:*

$$\prod_{j=1}^k \varepsilon_{j, \varphi^{\sigma(j)-1}(n)} \leq \eta_\sigma \prod_{j=1}^k \varepsilon_{j, \varphi^{j-1}(n)} \quad (3.3)$$

where $\eta_\sigma = \frac{1}{(k+1)!}$ for $\sigma \neq \text{Id}$, and $\eta_{\text{Id}} = 1$.

Proof of Lemma 3: First of all, let us notice that assumption (3.2) implies:

$$\forall i, j \in \{1, \dots, k\} \text{ with } i < j, \forall n \geq n_0, \forall n' \geq \varphi(n) \quad \varepsilon_{i,n'} \varepsilon_{j,n} \leq \frac{1}{(k+1)!} \varepsilon_{i,n} \varepsilon_{j,n'}. \quad (3.4)$$

For $\sigma \neq \text{Id}$ let κ_σ denote the largest integer $j \in \{1, \dots, k\}$ such that $\sigma(j) \neq j$; put also $\kappa_{\text{Id}} = 0$. We are going to prove Eq. (3.3) by induction on κ_σ . If $\kappa_\sigma \leq 1$ then $\sigma = \text{Id}$, so that Eq. (3.3) holds trivially. Let $\sigma \in \mathfrak{S}_k$ be such that $\kappa_\sigma \geq 2$, and assume that Eq.

(3.3) holds for any σ' such that $\kappa_{\sigma'} < \kappa_\sigma$. We have $\sigma(j) = j$ for any $j \in \{\kappa_\sigma + 1, \dots, k\}$, and $\sigma(\kappa_\sigma) < \kappa_\sigma$. Let $j_0 = \sigma^{-1}(\kappa_\sigma)$; then $j_0 < \kappa_\sigma$. Let $\sigma' = \sigma \circ \tau_{j_0, \kappa_\sigma}$ where $\tau_{j_0, \kappa_\sigma}$ is the transposition that exchanges j_0 and κ_σ . Then $\sigma'(j) = j$ for any $j \in \{\kappa_\sigma, \dots, k\}$ so that $\kappa_{\sigma'} < \kappa_\sigma$ and Eq. (3.3) holds for σ' . Since $\sigma'(j) = \sigma(j)$ for $j \notin \{j_0, \kappa_\sigma\}$, $\sigma'(j_0) = \sigma(\kappa_\sigma)$ and $\sigma'(\kappa_\sigma) = \kappa_\sigma$, this implies (using the fact that $\eta_{\sigma'} \leq 1$)

$$\varepsilon_{j_0, \varphi^{\sigma(\kappa_\sigma)-1}(n)} \varepsilon_{\kappa_\sigma, \varphi^{\kappa_\sigma-1}(n)} \prod_{\substack{1 \leq j \leq k \\ j \notin \{j_0, \kappa_\sigma\}}} \varepsilon_{j, \varphi^{\sigma(j)-1}(n)} \leq \prod_{j=1}^k \varepsilon_{j, \varphi^{j-1}(n)}.$$

On the other hand, Eq. (3.4) implies

$$\varepsilon_{j_0, \varphi^{\kappa_\sigma-1}(n)} \varepsilon_{\kappa_\sigma, \varphi^{\sigma(\kappa_\sigma)-1}(n)} \leq \frac{1}{(k+1)!} \varepsilon_{j_0, \varphi^{\sigma(\kappa_\sigma)-1}(n)} \varepsilon_{\kappa_\sigma, \varphi^{\kappa_\sigma-1}(n)}$$

because $j_0 < \kappa_\sigma$ and $\sigma(\kappa_\sigma) < \kappa_\sigma$ (so that $\varphi^{\sigma(\kappa_\sigma)-1}(n) < \varphi^{\kappa_\sigma-1}(n)$). Multiplying out the previous two inequalities yields Eq. (3.3) for σ , since $\sigma(j_0) = \kappa_\sigma$. This concludes the proof of Lemma 3.

Proof of Proposition 2: Let $n \geq n_0$. Denote by Δ the determinant of the matrix $[L_{\varphi^{i-1}(n)}(e_j)]_{1 \leq i, j \leq k}$. We have

$$|\Delta| \geq \prod_{j=1}^k \varepsilon_{j, \varphi^{j-1}(n)} - \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma \neq \text{Id}}} \prod_{j=1}^k \varepsilon_{j, \varphi^{\sigma(j)-1}(n)}$$

so that Lemma 3 applied to all $\sigma \neq \text{Id}$ yields

$$|\Delta| \geq \left(1 - \frac{1}{k+1}\right) \prod_{j=1}^k \varepsilon_{j, \varphi^{j-1}(n)}.$$

On the other hand, let $j \in \{1, \dots, k\}$ and Δ_j be the determinant of the matrix with columns C_1, \dots, C_k given by:

$$\begin{cases} C_{j'} = [L_{\varphi^{i-1}(n)}(e_{j'})]_{1 \leq i \leq k} \text{ for } j' \neq j, \\ C_j = [L_{\varphi^{i-1}(n)}(M)]_{1 \leq i \leq k}. \end{cases}$$

Since $M = \lambda_1 e_1 + \dots + \lambda_k e_k$, the lower bound proved for $|\Delta|$ yields:

$$|\Delta_j| = |\lambda_j| \cdot |\Delta| \geq \frac{k}{k+1} |\lambda_j| \prod_{j'=1}^k \varepsilon_{j', \varphi^{j'-1}(n)}. \quad (3.5)$$

Now expanding Δ_j with respect to the j -th column yields

$$|\Delta_j| \leq \sum_{i=1}^k |L_{\varphi^{i-1}(n)}(M)| \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma(j)=i}} \prod_{\substack{1 \leq j' \leq k \\ j' \neq j}} \varepsilon_{j', \varphi^{\sigma(j')-1}(n)}$$

so that, using Lemma 3:

$$|\Delta_j| \leq \left(\prod_{j'=1}^k \varepsilon_{j', \varphi^{j'-1}(n)} \right) \sum_{i=1}^k \frac{|L_{\varphi^{i-1}(n)}(M)|}{\varepsilon_{j, \varphi^{i-1}(n)}} \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma(j)=i}} \eta_\sigma.$$

Now we have $\eta_\sigma = 1$ for at most one σ , and $\eta_\sigma = \frac{1}{(k+1)!}$ for all other permutations σ among the $(k-1)!$ such that $\sigma(j) = i$, so that

$$\sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma(j)=i}} \eta_\sigma \leq 1 + \frac{(k-1)!}{(k+1)!} = \frac{k+1+\frac{1}{k}}{k+1},$$

thereby completing the proof of the upper bound

$$|\Delta_j| \leq \frac{k+1+\frac{1}{k}}{k+1} \left(\prod_{j'=1}^k \varepsilon_{j', \varphi^{j'-1}(n)} \right) \sum_{i=1}^k \frac{|L_{\varphi^{i-1}(n)}(M)|}{\varepsilon_{j, \varphi^{i-1}(n)}}.$$

Comparing this with the lower bound (3.5) concludes the proof of Proposition 2.

3.5 Proof of (iii)

We are now in position to prove the remaining part of Theorem 3, namely (iii). We assume $\tau_1 > \dots > \tau_k > 0$ and $\omega_1 = \dots = \omega_k = 0$ (see §3.1), so that $|L_n(e_j)| = Q_n^{-\tau_j+o(1)}$.

Let $\varepsilon > 0$. We choose $\varepsilon_1 > 0$ sufficiently small, so that

$$((1 + \varepsilon_1)^{k-1} - 1)\tau_1 < \varepsilon/4 \text{ and } (1 + \varepsilon_1)^{k-1} \leq 1 + \varepsilon/2. \quad (3.6)$$

If $k = 1$ there is no assumption on ε_1 , because it does not really appear in the proof: Proposition 2 is a triviality in this case, and the proof essentially reduces to that of [12].

For any $n \geq 1$, we define $\varphi(n)$ by $Q_{\varphi(n)-1} \leq Q_n^{1+\varepsilon_1} < Q_{\varphi(n)}$, because the sequence (Q_n) is increasing and we may assume $Q_n \geq 1$ for any n . Then we have $\varphi(n) \geq n+1$. This implies $\lim_{n \rightarrow +\infty} \varphi(n) = +\infty$, so that $Q_{\varphi(n)} = Q_{\varphi(n)-1}^{1+o(1)}$ (because we assume $Q_{n+1} = Q_n^{1+o(1)}$) and

$$Q_{\varphi(n)} = Q_n^{1+\varepsilon_1+o(1)}; \quad (3.7)$$

here $o(1)$ denotes any sequence that tends to 0 as $n \rightarrow \infty$. Letting $\varepsilon_{j,n} = |L_n(e_j)|$ for any $j \in \{1, \dots, k\}$ and any $n \geq 1$, the assumption $\varepsilon_{j,n} = Q_n^{-\tau_j+o(1)}$ implies $\varepsilon_{j,n} \neq 0$ for any j, n with n sufficiently large. Let $i \in \{1, \dots, k-1\}$, n_0 be sufficiently large, and let $n \geq n_0$, $n' \geq \varphi(n)$. Then we have $\varepsilon_{i,n'} = Q_{n'}^{-\tau_i+o(1)}$ (and the analogous property for $\varepsilon_{i+1,n'}$), where $o(1)$ denotes a sequence that tends to 0 as $n \rightarrow +\infty$, because we assume that $n' \geq \varphi(n) \geq n+1$. Using Eq. (3.7) and the assumption $\tau_i > \tau_{i+1}$ we obtain

$$\frac{\varepsilon_{i,n'} \varepsilon_{i+1,n}}{\varepsilon_{i,n} \varepsilon_{i+1,n'}} = \frac{Q_n^{\tau_i - \tau_{i+1} + o(1)}}{Q_{n'}^{\tau_i - \tau_{i+1} + o(1)}} \leq \frac{Q_n^{\tau_i - \tau_{i+1} + o(1)}}{Q_{\varphi(n)}^{\tau_i - \tau_{i+1} + o(1)}} = Q_n^{-(\tau_i - \tau_{i+1})\varepsilon_1 + o(1)} \leq \frac{1}{(k+1)!}$$

if n_0 is sufficiently large, so that assumption (3.2) of Proposition 2 holds.

Let Q be sufficiently large in terms of ε . Let $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ be such that $|\lambda_j| \leq Q^{\tau_j}$ for any j , and let $u \in (\text{Span}_{\mathbb{R}}(e_1, \dots, e_k))^\perp$ with $\|u\| \leq Q^{-1-\varepsilon}$. Assume that the point $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u$ belongs to $\mathbb{Z}^p \setminus \{(0, \dots, 0)\}$.

Let n be the least integer such that:

$$\text{for any } n' \geq n \text{ and any } j \in \{1, \dots, k\}, \text{ we have } |L_{n'}(e_j)| \leq \frac{Q^{-\tau_j}}{2k}.$$

Of course n depends on Q , and n tends to ∞ as $Q \rightarrow \infty$. Therefore if $u_n = o(1)$, that is $u_n \rightarrow 0$ as $n \rightarrow \infty$, then u_n tends also to 0 as $Q \rightarrow \infty$.

By minimality of n we have for some $j \in \{1, \dots, k\}$:

$$Q_n^{-\tau_j+o(1)} = |L_n(e_j)| \leq \frac{Q^{-\tau_j}}{2k} < |L_{n-1}(e_j)| = Q_{n-1}^{-\tau_j+o(1)} = Q_n^{-\tau_j+o(1)}$$

so that $Q = Q_n^{1+o(1)}$.

Let ℓ denote the least integer such that

$$\text{for any } \ell' \geq \ell \text{ and any } j \in \{1, \dots, k\}, \text{ we have } |\lambda_j L_{\ell'}(e_j)| \leq \frac{1}{2k}. \quad (3.8)$$

Since this upper bound holds for any $\ell' \geq n$, this integer exists and we have $\ell \leq n$.

The integer ℓ depends on Q and on the choice of $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u \in \mathbb{Z}^p \setminus \{(0, \dots, 0)\}$. Let us prove that $\ell \rightarrow \infty$ as $Q \rightarrow \infty$, independently on the choice of P . Let $\ell_0 \geq 1$, and K_{ℓ_0} denote the set of all points $M = \lambda'_1 e_1 + \dots + \lambda'_k e_k$ with $|\lambda'_j| \leq \frac{1}{2k|L_{\ell_0}(e_j)|}$ for any $j \in \{1, \dots, k\}$ (by making ℓ_0 larger if necessary, we may assume $L_{\ell_0}(e_j) \neq 0$). Then assertion (ii) proved in §3.2 yields $K_{\ell_0} \cap (\mathbb{Z}^p \setminus \{(0, \dots, 0)\}) = \emptyset$. Since K_{ℓ_0} is compact and $\mathbb{Z}^p \setminus \{(0, \dots, 0)\}$ is discrete, there exists $\eta > 0$ such that the distance of any point of K_{ℓ_0} to any point of $\mathbb{Z}^p \setminus \{(0, \dots, 0)\}$ is greater than η . If Q is sufficiently large, namely such that $Q^{-1-\varepsilon} < \eta$, then for any choice of $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u \in \mathbb{Z}^p \setminus \{(0, \dots, 0)\}$ with $\|u\| \leq Q^{-1-\varepsilon} < \eta$ we have $\lambda_1 e_1 + \dots + \lambda_k e_k \notin K_{\ell_0}$ so that $|\lambda_j L_{\ell_0}(e_j)| > \frac{1}{2k}$ for some $j \in \{1, \dots, k\}$. By definition of ℓ this implies $\ell > \ell_0$, and concludes the proof that $\ell \rightarrow \infty$ as $Q \rightarrow \infty$. In what follows, a sequence denoted by $o(1)$ will tend to 0 as n , ℓ or Q tends to ∞ ; therefore in any case, it tends to 0 as $Q \rightarrow \infty$.

Now, writing $u = (u_1, \dots, u_p)$ we have $\max_{1 \leq h \leq p} |u_h| \leq \|u\|$ so that for any $i \in \{1, \dots, k\}$:

$$\begin{aligned} |L_{\varphi^{i-1}(\ell)}(u)| &\leq \sum_{h=1}^p |u_h| |\ell_{h, \varphi^{i-1}(\ell)}| \leq p \|u\| Q_{\varphi^{i-1}(\ell)}^{1+o(1)} \\ &\leq p \|u\| Q_\ell^{(1+\varepsilon_1)^{i-1}(1+o(1))} \text{ using Eq. (3.7)} \\ &\leq p Q^{-1-\varepsilon} Q_n^{(1+\varepsilon_1)^{i-1}(1+o(1))} \text{ since } \ell \leq n \\ &\leq Q^{-1-\varepsilon+(1+\varepsilon_1)^{i-1}(1+o(1))} \text{ since } Q = Q_n^{1+o(1)} \\ &\leq Q^{-\varepsilon/2} \text{ thanks to the choice of } \varepsilon_1. \end{aligned}$$

On the other hand, Eq. (3.8) yields for any $i \in \{1, \dots, k\}$:

$$|L_{\varphi^{i-1}(\ell)}(\sum_{j=1}^k \lambda_j e_j)| \leq \sum_{j=1}^k \frac{1}{2k} = \frac{1}{2}.$$

Therefore we obtain for the point $P = \lambda_1 e_1 + \dots + \lambda_k e_k + u \in \mathbb{Z}^p$:

$$|L_{\varphi^{i-1}(\ell)}(P)| \leq Q^{-\varepsilon/2} + \frac{1}{2} < 1.$$

Now $L_{\varphi^{i-1}(\ell)}(P)$ is an integer for any $i \in \{1, \dots, k\}$, so it is zero. This yields the following upper bound on $|L_{\varphi^{i-1}(\ell)}(M)|$ (where we let $M = \sum_{j=1}^k \lambda_j e_j$):

$$|L_{\varphi^{i-1}(\ell)}(M)| = |L_{\varphi^{i-1}(\ell)}(u)| \leq Q^{-\varepsilon/2}.$$

Combining this upper bound with Proposition 2 yields, for any $j \in \{1, \dots, k\}$:

$$\begin{aligned} |\lambda_j L_{\ell-1}(e_j)| &\leq \left(1 + \frac{1}{k} + \frac{1}{k^2}\right) \sum_{i=1}^k Q^{-\varepsilon/2} Q^{\tau_j + o(1)} Q_{\varphi^{i-1}(\ell)}^{-\tau_j + o(1)} \\ &\leq Q_{\ell}^{\tau_j((1+\varepsilon_1)^{i-1}-1)+o(1)} Q^{-\varepsilon/2} \text{ using Eq. (3.7)} \\ &\leq Q_{\ell}^{\varepsilon/4+o(1)} Q^{-\varepsilon/2} \text{ using the assumption } \tau_j \leq \tau_1 \text{ and Eq. (3.6)} \\ &\leq Q_{\ell}^{-\varepsilon/4+o(1)} \leq \frac{1}{2k} \text{ since } Q_{\ell} \leq Q_n = Q^{1+o(1)}. \end{aligned}$$

This contradicts the minimality of ℓ in Eq. (3.8), thereby concluding the proof of (iii).

4 Application to zeta values

4.1 Construction of the linear forms

Let a, r, n be positive integers such that a is odd and $6r \leq a$. Let

$$S_n = (2n)!^{a-6r} \sum_{t=n+1}^{\infty} \frac{(t - (2r+1)n)_{2rn}^3 (t+n+1)_{2rn}^3}{(t-n)_{2n+1}^a}$$

and

$$S_n'' = \frac{1}{2} (2n)!^{a-6r} \sum_{t=n+1}^{\infty} \frac{d^2}{dt^2} \left(\frac{(t - (2r+1)n)_{2rn}^3 (t+n+1)_{2rn}^3}{(t-n)_{2n+1}^a} \right)$$

where the derivative is taken at t . As usual, Pochhammer's symbol is defined by $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$.

Letting $k = t - n$, we have obviously

$$S_n = (2n)!^{a-6r} \sum_{k=1}^{\infty} \frac{(k - 2rn)_{2rn}^3 (k + 2n + 1)_{2rn}^3}{(k)_{2n+1}^a}$$

(where the sum actually starts at $k = 2rn + 1$), and a similar expression for S_n'' .

It is well known (see for instance Théorème 5 of [10], Lemme 1 of [2] or Lemma 1.1 of [26]) that we have

$$S_n = \ell_{0,n} + \ell_{3,n}\zeta(3) + \ell_{5,n}\zeta(5) + \dots + \ell_{a,n}\zeta(a)$$

and

$$S_n'' = \ell_{0,n}'' + \ell_{3,n}\binom{4}{2}\zeta(5) + \ell_{5,n}\binom{6}{2}\zeta(7) + \dots + \ell_{a,n}\binom{a+1}{2}\zeta(a+2)$$

with $\ell_{0,n}, \ell_{0,n}'', \ell_{3,n}, \ell_{5,n}, \dots, \ell_{a,n} \in \mathbb{Q}$. Moreover d_{2n}^{a+2} is a common denominator of these rational numbers (see [26], Lemma 1.4), where d_k is the least common multiple of $1, 2, \dots, k$. Recall (for ulterior use) that $d_k = e^{k+o(k)}$ as $k \rightarrow \infty$, an equivalent form of the Prime Number Theorem.

We shall also need an upper bound on the coefficients of the linear forms S_n and S_n'' , namely

$$\max\left(|\ell_{0,n}|, |\ell_{0,n}'', |\ell_{3,n}|, |\ell_{5,n}|, \dots, |\ell_{a,n}|\right) \leq \left[2^{2(a-6r)}(2r+1)^{6(2r+1)}\right]^{n+o(n)}$$

as $n \rightarrow \infty$ (see Proposition 3.1 of [26]).

The asymptotic behaviour of $|S_n|$ and $|S_n''|$ is more subtle. From a technical point of view, this is the main difference with T. and K. Hessami-Pilehrood's construction [17] involving analogues of S_n and S_n'' : as $n \rightarrow \infty$, S_n'' oscillates whereas their analogue of S_n'' doesn't (see Lemma 6 of [17]). We state in the following lemma the properties we need, and postpone its proof until §4.3.

Lemma 4. *Assume that a is sufficiently large, and r is the integer part of $a \exp(-\sqrt{\log a})$. Then we have, as $n \rightarrow \infty$:*

$$|S_n| = \varepsilon_a^{n+o(n)} \text{ and } |S_n''| = \varepsilon_a^{n+o(n)} |\cos(n\omega_a + \varphi_a)|$$

with

$$0 < \varepsilon_a'' < \varepsilon_a \leq \frac{2^{6(r+1)}}{r^{2(a-6r)}} < 1 \text{ and } \varphi_a \not\equiv \frac{\pi}{2} \pmod{\pi}.$$

The strict inequality $\varepsilon_a'' < \varepsilon_a$ is very important for applying Theorem 3 (see §4.2), because it yields $\tau_1 \neq \tau_2$. This point is the main difficulty when trying to generalize Theorem 1 to vectors in \mathbb{R}^k involving $\zeta(j), \zeta(j+2), \dots, \zeta(j+2(k-1))$ for odd $j \leq a$.

4.2 Proof of the Diophantine results

To prove Theorem 1, we apply Theorem 3 to the linear forms $d_{2n}^{a+2}S_n$ and $d_{2n}^{a+2}S_n''$ where a is sufficiently large, and r is the integer part of $a \exp(-\sqrt{\log a})$. We use the following parameters: $k = 2$, $e_1 = (1, 0, \zeta(3), \zeta(5), \dots, \zeta(a))$, $e_2 = (0, 1, \binom{4}{2}\zeta(5), \binom{6}{2}\zeta(7), \dots, \binom{a+1}{2}\zeta(a+2))$, $Q_n = \beta^n$ with

$$\beta = e^{2(a+2)} 2^{2(a-6r)} (2r+1)^{6(2r+1)},$$

$\omega_1 = \varphi_1 = 0$, $\omega_2 = \omega_a$, $\varphi_2 = \varphi_a$, $\tau_1 = \frac{-\log(e^{2(a+2)}\varepsilon_a)}{\log \beta}$, $\tau_2 = \frac{-\log(e^{2(a+2)}\varepsilon_a'')}{\log \beta}$. This gives the following lower bound for the rank of the family in Theorem 1 (using Lemmas 1 and 4), as $a \rightarrow \infty$:

$$2 + \tau_1 + \tau_2 \geq \frac{-2}{\log \beta} \log \left(\frac{e^{2(a+2)} 2^{6(r+1)}}{r^{2(a-6r)}} \right) = \frac{4a \log r}{2a(1 + \log 2)} (1 + o(1)) = \frac{2 \log a}{1 + \log 2} (1 + o(1))$$

since $r \log r = o(\log a)$ and $\log r = (\log a)(1 + o(1))$. This concludes the proof of Theorem 1, except for Lemma 4 which we shall prove in the next subsection.

Let us deduce Corollary 1 from Theorem 1. Let $\tilde{n}_2(a)$ denote the number of odd integers $i \in \{3, 5, \dots, a\}$ such that $\zeta(i) \in \mathbb{Q}$ and $\zeta(i+2) \notin \mathbb{Q}$; then we have $\tilde{n}_2(a) = n_2(a)$ or $\tilde{n}_2(a) = n_2(a) + 1$. Moreover $n_1(a) + \tilde{n}_2(a)$ is the number of odd $i \in \{3, 5, \dots, a\}$ such that $\zeta(i)$ or $\zeta(i+2)$ is irrational. Since the vector $\begin{pmatrix} \zeta(i) \\ \binom{i+1}{2} \zeta(i+2) \end{pmatrix}$ can be removed from the family in Theorem 1 as soon as both $\zeta(i)$ and $\zeta(i+2)$ are rational numbers, the rank involved in this result is less than or equal to $n_1(a) + \tilde{n}_2(a) + 2 \leq n_1(a) + n_2(a) + 3$. Therefore Corollary 1 follows from Theorem 1.

To prove Theorem 2, we apply Theorem 3 with $k = 1$, $\omega_1 = \varphi_1 = 0$ (that is, Nesterenko's original linear independence criterion) to the linear forms $S_n + \lambda S_n''$. We take $Q_n = \beta^n$ as above, and $\tau = \tau_1 = \frac{-\log(e^{2(a+2)}\varepsilon_a)}{\log \beta}$ because $\varepsilon_a'' < \varepsilon_a$. This yields $\tau + 1 \geq \frac{\log a}{1 + \log 2} (1 + o(1))$ as a lower bound for the dimension, thereby proving Theorem 2.

4.3 Asymptotic properties

We now prove Lemma 4 stated in §4.1; this completes the proofs given in the previous subsection. The most difficult point (from a technical point of view) is the asymptotic behaviour of $|S_n''|$. It has been determined by Zudilin (§2 of [26]) in a general setting, using the saddle point method. We adapt this proof here, because we have to choose different parameters. Our notation is essentially taken from [26].

We keep the notation and assumptions of §4.1 (except for now that of Lemma 4); for instance we assume that a is odd and $6r \leq a$. We shall use the following polynomial:

$$Q(X) = (X + 2r + 1)^3 (X - 1)^{a+3} - (X - 2r - 1)^3 (X + 1)^{a+3} \in \mathbb{Q}[X]. \quad (4.1)$$

The complex roots of Q are localized in [26]; we are specially interested in two of them, denoted by τ_0 and μ_1 . The root τ_0 (resp. μ_1) is the unique root of Q in the domain $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$ (resp. $z \in \mathbb{R}$, $z > 2r + 1$) of \mathbb{C} . Of course Q , τ_0 and μ_1 depend on a and r (but not on n).

Let us consider the complex plane with cuts along the rays $(-\infty, 1]$ and $[2r + 1, +\infty)$. We let for $\tau \in \mathbb{C} \setminus ((-\infty, 1] \cup [2r + 1, +\infty))$:

$$\begin{aligned} f(\tau) &= 3(\tau + 2r + 1) \log(\tau + 2r + 1) + 3(2r + 1 - \tau) \log(2r + 1 - \tau) \\ &\quad + (a + 3)(\tau - 1) \log(\tau - 1) - (a + 3)(\tau + 1) \log(\tau + 1) + 2(a - 6r) \log(2). \end{aligned} \quad (4.2)$$

In this formula all logarithms are evaluated at positive real numbers if τ belongs to the real interval $(1, 2r+1)$, and we choose the determinations so that all of them take real values in this case. We also let, for $\tau \in \mathbb{C} \setminus ((-\infty, 1] \cup [2r+1, +\infty))$,

$$f_0(\tau) = f(\tau) - \tau f'(\tau).$$

For $\tau \in (2r+1, +\infty)$, let us denote by $\tau + i0$ the corresponding point on the upper bank of the cut $[2r+1, +\infty)$. Since $\log(2r+1 - (\tau + i0)) = \log(\tau - (2r+1)) - i\pi$ with $\tau - (2r+1) > 0$, we have for $\tau \in (2r+1, +\infty)$:

$$\begin{aligned} f(\tau + i0) &= 3(\tau + 2r + 1) \log(\tau + 2r + 1) + 3(2r + 1 - \tau) \log(\tau - 2r - 1) \\ &\quad + (a + 3)(\tau - 1) \log(\tau - 1) - (a + 3)(\tau + 1) \log(\tau + 1) + 2(a - 6r) \log(2) - 3(2r + 1 - \tau)i\pi. \end{aligned}$$

This function of $\tau \in (2r+1, +\infty)$ is increasing on $(2r+1, \mu_1)$, assumes a maximal value at $\tau = \mu_1$, and is decreasing on $(\mu_1, +\infty)$ (see Eq. (2.34) and Corollary 2.2 of [26]). Following the second proof of Lemma 3 in [2], we obtain:

$$\lim_{n \rightarrow \infty} \frac{\log |S_n|}{n} = \operatorname{Re} f(\mu_1 + i0) = \operatorname{Re} f_0(\mu_1 + i0) \quad (4.3)$$

since $f'(\mu_1 + i0) = 3i\pi \in i\mathbb{R}$. The main difference with the proof of [2] is the term $2(a - 6r) \log 2$ in (4.2), which comes from the fact that the integer denoted here by n is actually $2n$ with the notation of [2]; this has an effect because of the normalization factor $(2n)!^{a-6r}$.

By applying the saddle point method, Zudilin proves the following result, where τ_0 is defined (as above) to be unique root of Q in the domain $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$, and g is defined on the cut plane $\mathbb{C} \setminus ((-\infty, 1] \cup [2r+1, +\infty))$ by

$$g(\tau) = \frac{(\tau + 2r + 1)^{3/2} (2r + 1 - \tau)^{3/2}}{(\tau + 1)^{(a+3)/2} (\tau - 1)^{(a+3)/2}}. \quad (4.4)$$

Lemma 5. *Assume that a is odd, $6r \leq a$, and*

$$\mu_1 \leq 2r + 1 + \min \left(\frac{3r(r+1)}{2(a+3)}, \frac{r(r+1)}{3(2r+1)} \right). \quad (4.5)$$

Let $\varepsilon''_{r,a} = \exp \operatorname{Re} f_0(\tau_0)$, $\omega_{r,a} = \operatorname{Im} f_0(\tau_0)$, and $\varphi_{r,a} = -\frac{1}{2} \arg f''(\tau_0) + \arg g(\tau_0)$. If

$$\text{either } \varphi_{r,a} \not\equiv \frac{\pi}{2} \pmod{\pi} \text{ or } \omega_{r,a} \not\equiv 0 \pmod{\pi} \quad (4.6)$$

then we have, as $n \rightarrow \infty$,

$$|S''_n| = \varepsilon''_{r,a}{}^{m+o(n)} |\cos(n\omega_{r,a} + \varphi_{r,a}) + o(1)|.$$

We shall now deduce Lemma 4 from Lemma 5, thereby concluding the proof of Theorem 1. With this aim in mind, we assume from now on that a is sufficiently large, and r is the integer part of $a \exp(-\sqrt{\log a})$. To begin with, let us check the assumptions of Lemma 5 in this case. From now on we denote by $o(1)$ any sequence that depends only on a and tends to 0 as $a \rightarrow \infty$.

The polynomial Q defined in (4.1) satisfies $Q(2r+1) > 0$. Let $\nu(a) = \exp(-\exp \sqrt[3]{\log a})$; then we have also

$$Q(2r+1+\nu(a)) = (2r)^{a+3} \left[\left(4r+2+\nu(a)\right)^3 \left(1+\frac{\nu(a)}{2r}\right)^{a+3} - \nu(a)^3 \left(1+\frac{1}{r}+\frac{\nu(a)}{2r}\right)^{a+3} \right] < 0$$

provided a is sufficiently large, because (since $\log r = o(a/r)$)

$$3 \log(4r+2+\nu(a)) + (a+3) \log\left(1+\frac{\nu(a)}{2r}\right) \leq 4 \log(r) + \frac{(a+3)\nu(a)}{2r} \leq \frac{a+3}{6r},$$

and also, since $|\log \nu(a)| = o(a/r)$:

$$3 \log \nu(a) + (a+3) \log\left(1+\frac{1}{r}+\frac{\nu(a)}{2r}\right) \geq 3 \log \nu(a) + \frac{a+3}{2r} \geq \frac{a+3}{4r}.$$

Therefore the root μ_1 of Q (which is the only one in the real interval $(2r+1, +\infty)$) satisfies $2r+1 < \mu_1 < 2r+1+\nu(a)$. As a corollary, the assumption (4.5) of Lemma 5 holds. Moreover we have $|\tau_0 - 2r - 1| < |\mu_1 - 2r - 1|$ (see the proof of Lemma 2.7 (e) of [26]), so that

$$|\tau_0 - 2r - 1| < \nu(a). \quad (4.7)$$

This upper bound will be used now to check assumption (4.6) of Lemma 5. Following the notation of [26] we let

$$\alpha_+ = \arg(\tau_0 - 1), \alpha_- = \arg(\tau_0 + 1), \beta_+ = -\arg(2r+1-\tau_0), \beta_- = \arg(\tau_0 + 2r+1),$$

where these arguments belong to $(-\pi, \pi)$ and would be 0 if τ_0 were in the real interval $(1, 2r+1)$. Then Eq. (2.26) in the proof of Lemma 2.7 of [26] reads (with $\tau = \tau_0$ and $f'(\tau_0) = i\pi$):

$$3(\beta_- + \beta_+) + (a+3)(\alpha_+ - \alpha_-) = \pi. \quad (4.8)$$

Now Eq. (4.7) yields

$$0 \leq \sin \alpha_+ = \frac{\operatorname{Im} \tau_0}{|\tau_0 - 1|} \leq \frac{|\tau_0 - (2r+1)|}{|\tau_0 - 1|} \leq \frac{\nu(a)}{r}$$

so that $a\alpha_+$ tends to 0 as $a \rightarrow \infty$. The same property can be proved in the same way for α_- and β_- , so that Eq. (4.8) yields $\beta_+ = \pi/3 + o(1)$.

Now let us deduce that assumption (4.6) of Lemma 5 holds. We have

$$f''(\tau_0) = \frac{3}{\tau_0 + 2r + 1} + \frac{3}{2r + 1 - \tau_0} + \frac{a+3}{\tau_0 - 1} - \frac{a+3}{\tau_0 + 1}$$

(see at the end of the proof of Lemma 2.9 of [26]), so that $f''(\tau_0) = \frac{3(1+o(1))}{2r+1-\tau_0}$ and

$$\arg f''(\tau_0) \equiv -\arg(2r+1-\tau_0) + o(1) \equiv \beta_+ + o(1) \equiv \pi/3 + o(1) \pmod{2\pi}.$$

In the same way the definition of g yields

$$\arg g(\tau_0) \equiv \frac{3}{2}\beta_- - \frac{3}{2}\beta_+ - \frac{a+3}{2}\alpha_- - \frac{a+3}{2}\alpha_+ \equiv -\frac{\pi}{2} + o(1) \pmod{2\pi}$$

so that

$$-\frac{1}{2} \arg f''(\tau_0) + \arg g(\tau_0) \equiv \frac{-2\pi}{3} + o(1) \not\equiv \frac{\pi}{2} \pmod{\pi}$$

if a is sufficiently large, so that Eq. (4.6) holds.

We have proved that Lemma 5 applies when r is the integer part of $a \exp(-\sqrt{\log a})$ and a is sufficiently large. It provides $\varepsilon''_{r,a}$, $\omega_{r,a}$ and $\varphi_{r,a}$, which we denote respectively by ε''_a , ω_a and φ_a . We also let $\varepsilon_a = \lim_{n \rightarrow \infty} \frac{\log |S_n|}{n} = \operatorname{Re} f_0(\mu_1 + i0)$ (see Eq. (4.3)).

The inequality $\varepsilon''_a < \varepsilon_a$ follows from Lemma 2.10 of [26]. To prove that $\varepsilon_a \leq \frac{2^{6(r+1)}}{r^{2(a-6r)}}$, we follow [2] (end of the second proof of Lemme 3). In precise terms, for any $k > 2rn$ we have $k + (2r+1)n < 2^{1+1/r}k$, so that

$$\begin{aligned} (2n)!^{a-6r} \frac{(k-2rn)_{2rn}^3 (k+2n+1)_{2rn}^3}{(k+1)_{2n}^a} &< (2n)^{2n(a-6r)} \frac{k^{6rn} (2^{1+1/r}k)^{6rn}}{k^{2an}} \\ &= \left(\frac{2n}{k}\right)^{2n(a-6r)} 2^{6rn(1+1/r)} < \left[\frac{2^{6(r+1)}}{r^{2(a-6r)}}\right]^n. \end{aligned}$$

This yields

$$|S_n| \leq \left[\frac{2^{6(r+1)}}{r^{2(a-6r)}}\right]^n \sum_{k=2rn+1}^{+\infty} \frac{1}{k^a}$$

so that $\varepsilon_a \leq \frac{2^{6(r+1)}}{r^{2(a-6r)}}$. This concludes the proof of Lemma 4.

References

- [1] R. APÉRY – “Irrationalité de $\zeta(2)$ et $\zeta(3)$ ”, in *Journées Arithmétiques (Luminy, 1978)*, Astérisque, no. 61, 1979, p. 11–13.
- [2] K. BALL & T. RIVOAL – “Irrationalité d’une infinité de valeurs de la fonction zêta aux entiers impairs”, *Invent. Math.* **146** (2001), no. 1, p. 193–207.
- [3] F. BEUKERS – “Padé-approximations in number theory”, in *Padé approximation and its applications (Amsterdam, 1980)*, Lecture Notes in Math., no. 888, Springer, 1981, p. 90–99.
- [4] N. BOURBAKI – “Algèbre”, ch. II, Hermann, third. éd., 1962.

- [5] Y. BUGEAUD & M. LAURENT – “On transfer inequalities in Diophantine approximation, II”, *Math. Z.* **265** (2010), p. 249–262.
- [6] J. CASSELS – *An introduction to the geometry of numbers*, Grundlehren der Math. Wiss., no. 99, Springer, 1959.
- [7] N. FEL'DMAN & Y. NESTERENKO – *Number theory IV, transcendental numbers*, Encyclopaedia of Mathematical Sciences, no. 44, Springer, 1998, A.N. Parshin and I.R. Shafarevich, eds.
- [8] S. FISCHLER – “Nesterenko’s criterion when the small linear forms oscillate”, *Archiv der Math.*, to appear.
- [9] S. FISCHLER, M. HUSSAIN, S. KRISTENSEN & J. LEVESLEY – “A converse to linear independence criteria, valid almost everywhere”, work in progress.
- [10] S. FISCHLER & T. RIVOAL – “Approximants de Padé et séries hypergéométriques équilibrées”, *J. Math. Pures Appl.* **82** (2003), no. 10, p. 1369–1394.
- [11] — , “Irrationality exponent and rational approximations with prescribed growth”, *Proc. Amer. Math. Soc.* **138** (2010), no. 8, p. 799–808.
- [12] S. FISCHLER & W. ZUDILIN – “A refinement of Nesterenko’s linear independence criterion with applications to zeta values”, *Math. Ann.* **347** (2010), p. 739–763.
- [13] L. GUTNIK – “On the irrationality of some quantities containing $\zeta(3)$ ”, *Acta Arith.* **42** (1983), no. 3, p. 255–264, (in Russian) ; translation in *Amer. Math. Soc. Transl.* **140** (1988), p. 45–55.
- [14] — , “On linear forms with coefficients in $\mathbb{N}\zeta(1 + \mathbb{N})$ ”, in *Proceedings of the Session in analytic number theory and Diophantine equations (Bonn, 2002)* (D. Heath-Brown & B. Moroz, eds.), Bonner Mathematische Schriften, no. 360, 2003, p. 1–45.
- [15] M. HATA – “Rational approximations to π and some other numbers”, *Acta Arith.* **63** (1993), no. 4, p. 335–349.
- [16] — , “The irrationality of $\log(1 + 1/q) \log(1 - 1/q)$ ”, *Trans. Amer. Math. Soc.* **350** (1998), no. 6, p. 2311–2327.
- [17] T. HESSAMI PILEHROOD & K. HESSAMI PILEHROOD – “Irrationality of sums of zeta values”, *Mat. Zametki [Math. Notes]* **79** (2006), no. 4, p. 607–618 [561–571].
- [18] T. HESSAMI PILEHROOD – “Linear independence of vectors with polylogarithmic coordinates”, *Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.]* **54** (1999), no. 6, p. 54–56 [40–42].

- [19] M. LAURENT – “On transfer inequalities in Diophantine approximation”, in *Analytic Number Theory, Essays in Honour of Klaus Roth* (W. Chen, W. Gowers, H. Halberstam, W. Schmidt & R. Vaughan, eds.), Cambridge Univ. Press, 2009, p. 306–314.
- [20] R. MARCOVECCHIO – “Linear independence of linear forms in polylogarithms”, *Annali Scuola Norm. Sup. Pisa* **V** (2006), no. 1, p. 1–11.
- [21] Y. NESTERENKO – “On the linear independence of numbers”, *Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.]* **40** (1985), no. 1, p. 46–49 [69–74].
- [22] E. NIKISHIN – “On the irrationality of the values of the functions $F(x, s)$ ”, *Mat. Sbornik [Math. USSR-Sb.]* **109** [37] (1979), no. 3, p. 410–417 [381–388].
- [23] T. RIVOAL – “La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs”, *C. R. Acad. Sci. Paris, Ser. I* **331** (2000), no. 4, p. 267–270.
- [24] —, “Indépendance linéaire des valeurs des polylogarithmes”, *J. Théor. Nombres Bordeaux* **15** (2003), no. 2, p. 551–559.
- [25] W. SCHMIDT – “On heights of algebraic subspaces and Diophantine approximations”, *Annals of Math.* **85** (1967), p. 430–472.
- [26] W. ZUDILIN – “Irrationality of values of the Riemann zeta function”, *Izvestiya Ross. Akad. Nauk Ser. Mat. [Izv. Math.]* **66** (2002), no. 3, p. 49–102 [489–542].

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